MAT 685: C++ for Mathematicians

Prime pairs in Monte Carlo

John Perry

University of Southern Mississippi

Spring 2017

Outline

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Classic problem

- Choose n
- Choose $a, b \in \{1, \ldots, n\}$
- Let p_n be probability that $\gcd(a, b) = 1$
- Does $\lim_{n \to \infty} p_n$ exist?
  - If so, what is its value?

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Example

Let $n = 8$.

- Possible outcomes: $\binom{8}{2} = \frac{8!}{2!6!} = 28$
- Relatively prime pairs: $(1,2)$, $(1,3)$, ..., $(1,8)$, $(2,3)$, $(2,5)$, $(2,7)$, $(3,4)$, $(3,5)$, $(3,7)$, $(3,8)$, $(4,5)$, $(4,7)$, $(6,7)$, $(7,8)$
- So $p_8 = {}^{18}/_{28} = 9/14 \approx 64.3\%$

A problem

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$n \to \infty$: program takes longer, and longer, and looooooongerrrrrrr...

Probabilistic algorithms

Two major types

Las Vegas "always correct, probably fast"

Monte Carlo "always fast, probably correct"

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Our approach

For large n:

- choose large subset of pairs
- determine how many are rel. prime
- divide by size of subset

Which algorithm type is this?

Example

$n = 100$

- Choose 10: (rel prime blue)

$(53, 97)$	$(22, 99)$	$(30, 42)$
$(20, 69)$	$(3, 9)$	$(61, 80)$
$(3, 37)$	$(9, 74)$	$(22, 43)$
	$(67, 94)$	

Example

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- estimate: 70%
- actual: 69%

Not bad!

(and much less work than $\frac{100 \cdot 101}{2}$)

Outline

How should we choose pairs?

- no obvious pattern
- "uniform" distribution

$\therefore$ **Choose randomly**

(of course)

Question

But how do we generate random numbers?!?

We don't. We generate *pseudo*-random numbers.

linear congruential generator $x_{n+1} = (ax_n + b) \bmod c$

seed x_0 (first number in sequence)

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Example

Let $a = 3, b = 7, c = 10, x_0 = 2$

$$2, 3, 6, 5, 2, 3, 6, 5, 2, 3, 6$$

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Example

Let $a = 3, b = 7, c = 10, x_0 = 2$

$$2, 3, 6, 5, 2, 3, 6, 5, 2, 3, 6$$

...not a very good choice!

C++ library helps out

- Pseudo-random generator

```
#include <cstdlib>
void srand(unsigned seed)
int rand()
```

  - use `srand()` to seed, usually to system time
  - `rand()` result in [`0`, `RAND_MAX`]

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- System time?

```
#include <ctime>
time_t time(time_t * storage)
```

  - `storage` is "address" of optional variable to store time
  - use `nullptr` for now

`nullptr`?

- C++11 standard for "nowhere"
- previous conventions: 0, `NULL`
  - conventions, not standards
  - **avoid these**

Book offers adaptation

Why?

- specify different interval
  - not just `[0, RAND_MAX]`
- ways to avoid patterns in certain special sets
- different generator desired? easier to change

Interface (p. 1)

New folder, `rand_unif`

Listing 1: `uniform.hpp` (p. 1)

```
#ifndef UNIFORM_HPP
#define UNIFORM_HPP

/** Generate a random number between 0 and 1.*/
double unif();

/**
  Generate a random number in a real interval.
  @param a one endpoint of the interval
  @param b the other endpoint of the interval
  @return a uniform random number in [0,1]
*/
double unif(double a, double b);
```

Interface (p. 2)

Listing 2: `uniform.hpp` (p. 2)

```
/**
  Generate a random integer between 1 and a
  given value.
  @param n the largest possible value
  @return uniform random value in {1,...,n}
*/
long unif(long n);

/** Reset generator based on system clock */
void seed();

#endif
```

Implementation (p. 1)

Listing 3: `uniform.cpp` (p. 1)

```
#include <ctime>
using std::time;
#include <cstdlib>
using std::rand; using std::srand;

#include "uniform.hpp"

double unif() {
  return rand() / double(RAND_MAX);
}

double unif(double a, double b) {
  return (b - a)*unif() + a;
}
```

Implementation (p. 2)

Listing 4: `uniform.cpp` (p. 2)

```
long unif(long a) {
  if (a == 0) return 0;
  if (a < 0) a = -a;
  return long(unif()*a) + 1;
}

void seed() {
  srand(time(nullptr));
}
```

Outline

Organizing files

- It is common to organize source files in folders and subfolders.
- Currently, though, we have a mess:

```
c++_main
  gcd
    gcd.hpp
    gcd_recursive.cpp
    gcd_while.cpp
    prob_relprime_pair.cpp
    test_gcd_recursive.cpp
    test_gcd_while.cpp
    test_xgcd.cpp
  rand_unif
    uniform.cpp
    uniform.hpp
```

How to access both sets of files?

Monte Carlo approach needs both gcd and uniform files. They are in different folders.

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- Create new folder **`relprime_pairs`**
  - move `prob_relprime_pair.cpp` in
  - no longer compiles; we can fix it

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  - move `prob_relprime_pair.cpp` in
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`#include "/path/to/filename"` can use absolute or relative paths to an interface

- change `#include ``gcd.hpp''` to `#include ``../gcd/gcd.hpp''`
- compilation:

```
$ cd relprime_pairs
$ g++ -o prob_relprime_pair \
    ../gcd/gcd_while.cpp \
    prob_relprime_pair.cpp
$ ./prob_relprime_pair
```

Homework

pp. 63–65 #4.1, 4.4, 4.5

Outline

Monte Carlo method

Place in `relprime_pairs` folder

Listing 5: `monte_carlo_pairs.cpp` (p. 1)

```
#include <iostream>
using std::cin; using std::cout;
using std::endl;

#include "../gcd/gcd.hpp"
#include "../rand_unif/uniform.hpp"

int main() {
  long n;
  long reps;  // # times we perform experiment
  long a, b;  // pairs in {1,..,n}
  long count; // number of rel prime pairs

  count = 0;
  cout << "Enter n (max el of set) --> ";
  cin >> n;
```

Monte Carlo method

Place in `relprime_pairs` folder

Listing 6: `monte_carlo_pairs.cpp` (p. 2)

```
  cout << "Enter number of pairs to sample --> ";
  cin >> reps;

  seed();
  for (long k = 0; k < reps; ++k) {
    a = unif(n);
    b = unif(n);
    if (gcd(a,b) == 1) ++count;
  }

  cout << count / double(reps) << endl;

  return 0;
}
```

Compiling, executing

Compiling... pay attention to dots!

```
$ g++ -o monte_carlo_pairs \
    -std=c++11 \
    ../gcd/gcd_while.cpp \
    ../rand_unif/uniform.o \
    monte_carlo_pairs.cpp
```

(may need `-std=c++11` to avoid complaint about `nullptr`)

Compiling, executing

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$ g++ -o monte_carlo_pairs \
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    monte_carlo_pairs.cpp
```

(may need `-std=c++11` to avoid complaint about `nullptr`)

Executing

```
$ ./monte_carlo_pairs
Enter n (max el of set) --> 100000
Enter number of pairs to sample --> 1000
0.595
```

Comparison

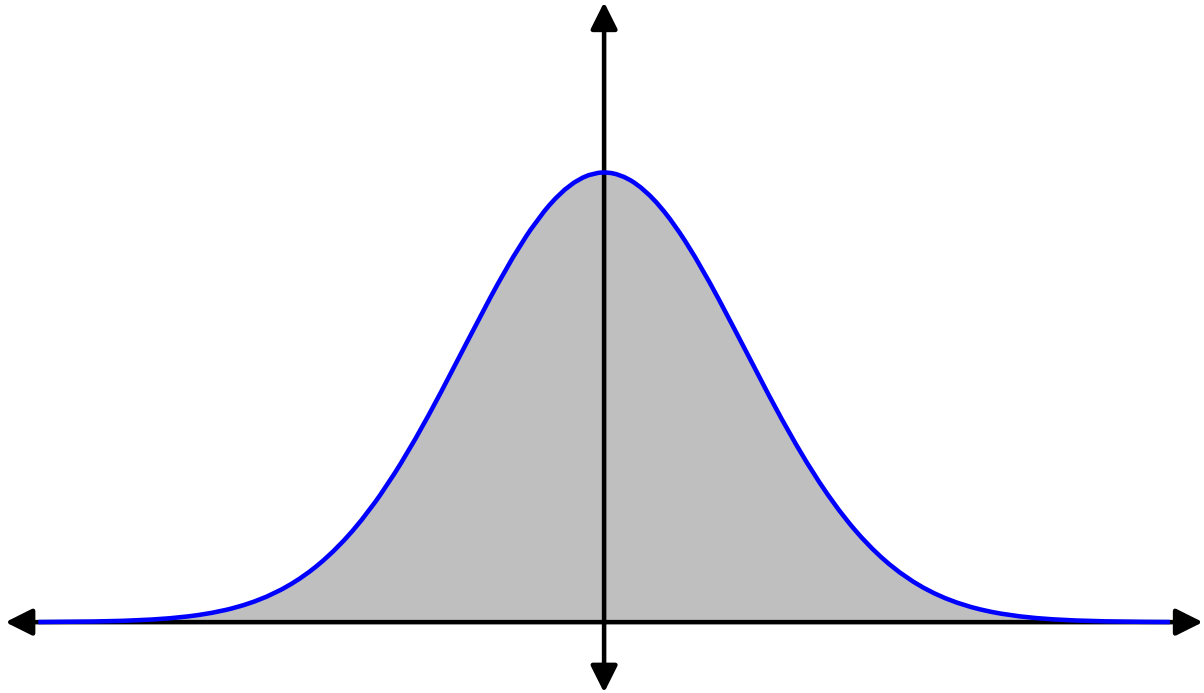
Estimate: 0.595
Actual: 0.614747

- `prob_relprime_pairs` takes *a long time* to find it
- can get different estimates, too:
  - different execution times
  - different values of `reps`
- can approximate correct value w/many estimates *faster* than finding exact

Outline

Not all distributions are uniform

"Normal" random variables are Gaussian:



$$f(t) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}}$$

Probabilistic properties

$$\int_{\mathbb{R}} f \, dt = 1 \quad \text{area under curve}$$
$$\int_{\mathbb{R}} tf \, dt = 0 \quad \text{mean}$$
$$\int_{\mathbb{R}} t^2 f \, dt = 1 \quad \text{standard deviation}$$

Probabilistic properties

$$\begin{array}{ll} \int_{\mathbb{R}} f\ dt = 1 & \text{area under curve} \\ \int_{\mathbb{R}} tf\ dt = 0 & \text{mean} \\ \int_{\mathbb{R}} t^2 f\ dt = 1 & \text{standard deviation} \end{array}$$

$$\text{define } \Phi(x) = P|X \leq x| = \int_{-\infty}^{x} f(t)\ dt$$

Generating normal random values

Box-Muller method

generate (x, y) uniformly in unit disc

let

$$r = \sqrt{x^2 + y^2}, \quad \mu = \sqrt{\frac{-2 \ln r}{r}}, \quad z = \mu x$$

return z

Generating normal random values

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generate (x, y) uniformly in unit disc
let

$$r = \sqrt{x^2 + y^2}, \quad \mu = \sqrt{\frac{-2 \ln r}{r}}, \quad z = \mu x$$

return z

Remark

$y = \mu y$ would also be random value

`static` "remembers" values

Listing 7: `memories.cpp`

```
#include <iostream>
using std::cin; using std::cout; using std::endl;

int memory(int val) {
  static int last_val; // static, so "remembers"
  int ret_val = last_val; // temporary
  last_val = val; // replace
  return ret_val; // return
}

int main() {
  for (int i = 0; i < 10; ++i) {
    int new_val;
    cout << "Enter #" << i << ": ";
    cin >> new_val;
    cout << "I remember " << memory(new_val) << endl;
  }
}
```

Compiling, executing

```
$ g++ -o memories memories.cpp
$ ./memories
Enter #0: 5
I remember 0
Enter #1: 3
I remember 5
Enter #2:
```

Compiling, executing

```
$ g++ -o memories memories.cpp
$ ./memories
Enter #0: 5
I remember 0
Enter #1: 3
I remember 5
Enter #2:
```

...now back to normal random values!

Interface

New folder, `rand_norm`

Listing 8: `randn.hpp`

```
#ifndef RANDN_HPP
#define RANDN_HPP

/**
  Generate a random number between -1 and 1
  according to a normal distribution.
*/
double randn();

#endif
```

Implementation

Listing 9: `randn.cpp` (p. 1)

```
#include <cmath>
using std::sqrt;

#include "../rand_unif/uniform.hpp"

double randn() {
  static bool has_saved = false;
  static double saved;

  // use saved value, if we have one
  if (has_saved) {
    has_saved = false;
    return saved;
  }

  // pick a point within unit disk
  double x, y;
  double r = 2.0;
```

Implementation

Listing 10: `randn.cpp` (p. 2)

```
  while (r > 1.0) {
    x = unif(-1.0, 1.0);
    y = unif(-1.0, 1.0);
    r = x*x + y*y;
  }

  // rescale by Box-Muller
  r = sqrt(r);
  double mu = sqrt(-2.0 * log(r) / r);

  // save y value, return x value
  saved = mu*y;
  has_saved = true;

  return mu*x;
}
```

Program to compare

Place in parent folder

Listing 11: `random_comparisons.cpp`

```
#include <iostream>
using std::cout; using std::endl;

#include "rand_unif/uniform.hpp"
#include "rand_norm/randn.hpp"

int main() {
  seed();

  cout << "Uniform:\n";
  for (int i = 0; i < 20; ++i)
    cout << unif(10) << ',';
  cout << endl;

  cout << "Normal:\n";
  for (int i = 0; i < 20; ++i)
    cout << long(randn()*10/4) + 5 << ',';
  cout << endl;
}
```

Compiling, executing

```
$ g++ -o random_comparisons rand_unif/uniform.o \
    rand_norm/randn.o random_comparisons.cpp
$ ./random_comparisons
Uniform:
9,4,8,8,10,2,4,8,3,6,5,7,4,6,10,10,7,8,2,7,
Normal:
5,5,4,5,5,7,6,3,8,5,5,5,4,6,5,7,5,6,3,4,
$
```

Compiling, executing

```
$ g++ -o random_comparisons rand_unif/uniform.o \
    rand_norm/randn.o random_comparisons.cpp
$ ./random_comparisons
Uniform:
9,4,8,8,10,2,4,8,3,6,5,7,4,6,10,10,7,8,2,7,
Normal:
5,5,4,5,5,7,6,3,8,5,5,5,4,6,5,7,5,6,3,4,
$
```

Notice bunching of normal

- 5,6,7 appear 14 times

Homework

p. 65 #4.6, 4.7

Outline

Summary

- Math stuff
  - Monte Carlo algorithms
  - Probability distributions
  - Pseudorandom generators
    - Linear congruential generator
    - Box-Muller method
- Programming stuff
  - C++ Standard Library: random
  - file organization
  - `static` variables