MAT 685: C++ for Mathematicians

A modular class

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Modular arithmetic

Observations from the code...

...on the language

...on the mathematics

Summary

Outline

1. Modular arithmetic
2. Observations from the code...
   - ...on the language
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From Modern Algebra or Number Theory

$\mathbb{Z}_n = \{0, 1, \ldots, m-1\}$

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- has properties of a **ring**
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  - multiplication works "almost as desired" (closed, commutative, associative, identity, *not always inverses*)

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  - multiplication works "almost as desired" (closed, commutative, associative, identity, *not always inverses*)
- if m prime, has properties of a **field**
  - addition, subtraction same
  - multiplication "as desired": inverses, too!

Very useful

- anything cyclical
- "clockwork" arithmetic: 3 hours after 10 is 1 ($10 + 3 = 1$)
- coding theory
- cryptography
- computer science

Let's implement as a class

- Constructors
  - include default constructor
    - default value obviously 0
    - default modulus? *not* 0

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- Constructors
  - include default constructor
    - default value obviously 0
    - default modulus? *not* 0
- Getters
- Setters
- I/O
- Arithmetic operators
- Comparison operators

Default modulus?

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  - annoying & wasteful to change from *our* default modulus for every new object
- why not both?

`constexpr` and `static`

`constexpr` expression can be evaluated to a constant at compile time (C++11 only)

- all involved expressions must be `constexpr` too
- appropriate choice for *our* default modulus
- observe different meaning from `const`, as book uses

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`static` field w/same value in *all* objects of class

- changing it in one object changes it in all
- can change w/out object by scoping: `Class::static_field = new_value`
- appropriate choice for *user's* default modulus

Possible problem

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Throw exception

exception special class for "bad data"

- `#include <stdexcept>`
- see available exceptions
- can create your own, too!

- terminates program immediately
- can provide useful information
- usage: **`throw`** `exception_type(``string description'');`

Notes on throwing exceptions

1. Do it rarely...
   - can slow down code
   - better to write good code from the get-go
2. ...but do it when necessary
   - undefined behavior hard to debug
   - client can `try` and `catch()` exceptions, allowing recovery

Interface

Ah, just look at the source code.

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Defined methods in class

Not typically done (or advisable)

- necessary for templates, inline code (see below)
- sometimes inadvisable: linker complaints?
- no need for class scope (`Mod::`) if in class

Can also define outside *class* but in *interface file*

- need full signature (class scope)

```
inline Mod Mod::pow (int64_t k) const {
  // a miracle occurs here
}
```

Inlining code

Function calls expensive

- involve small, but nonzero time penalty
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`inline` keyword *requests*, but does not *guarantee* inlining

- compiler free to ignore
- compiler can inline even when not requested
- many counsel against using this keyword, but...
- ...I've seen it make a difference

Static fields

```
static int64_t default_modulus;
```

- Universal to class: accessing `default_modulus` anywhere changes for all instances

x	y	z
val=7	val=4	val=8
mod=10	mod=11	mod=11
default_modulus=11		

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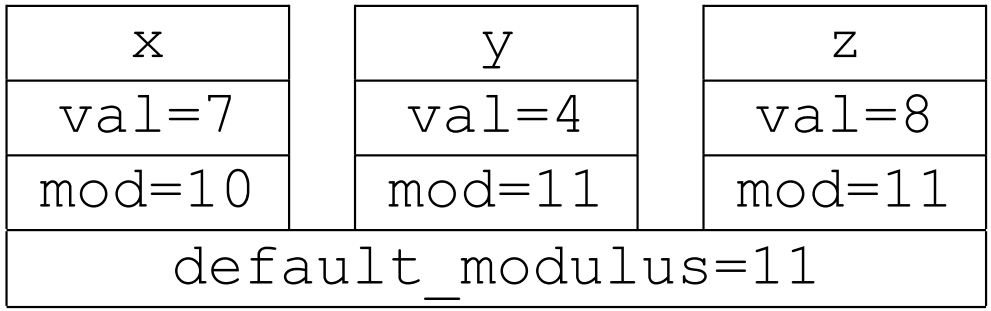

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solution: `static` functions

```
static int64_t Mod::get_default_modulus();
static void Mod::set_default_modulus(int64_t);
```

Initializing static fields

May not initialize non-constant static variable *in the class:*

```
static const int64_t default_modulus
    = INITIAL_DEFAULT_MODULUS; // cool!
static int64_t default_modulus
    = INITIAL_DEFAULT_MODULUS; // error!
```

`default_modulus` not constant, so this is out.

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May initialize *anywhere outside the class:*

```
class Mod {
public:
  static void set_default_modulus(int64_t m) {
    // a miracle occurs here
  }
};

Mod::set_default_modulus(INITIAL_DEFAULT_MODULUS);
```

Constructors and fields

```
Mod() : mod(default_modulus), val(0) { }

Mod(int64_t x)
    : mod(default_modulus), val(x) { adjust_val(); }

Mod(int64_t x, int64_t m) : val(x), mod(m) {
  check_mod(); adjust_val();
}

Mod(const Mod & other) : val(other.val), mod(other.mod)
{ }
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- In last three cases, `val` also initialized before code is run.
- In each case, *we construct before we compute.*

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- Sometimes this is *required* (`const` fields).
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- In each case, *we construct before we compute.*
- Sometimes this is *required* (`const` fields).
- It is typically *more efficient*.
- So: *construct before computing*

Assignment is sometimes construction

Test code has

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Interface does not define

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It *could* define it, but if not, corresponding constructor is called:

```
Mod::Mod(int64_t x);
```

Arithmetic operators

C++ assumes nothing: defining

```
Mod operator + (
    const Mod & m, const int64_t other
);
```

does not likewise define

```
Mod operator + (
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If you want both, you must define both.

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);
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If you want both, you must define both.
In our case,

```
Mod Mod::operator + (const int64_t other);
```

does the same as the first.

Arithmetic operator's return value

Standard practice is to return result of *any* operation, ***including* increment, decrement, scaling**

```
Mod & Mod::operator ++ () {
  ++val; adjust_val();
  return Mod(val, mod);
}
```

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}
```

Allows us to chain operations, an old C++ trick

```
cout << "++x = " << ++x << endl;
```

(Many languages do not allow this.)

Should we really check correct modulus?

In arithmetic operations we check `mod` & `throw()` if different, *but...*

- checking `mod` slows arithmetic
- arguably client's job, not ours

Pre- and post-inc/decrement

Pre-increment

- "increment before returning value"
- `symbol` before variable
- return new value
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Post-increment

- "increment after returning value"
- `symbol` after variable
- return old value
- signature: `Type Type::operator ++ (int) ;`
  - argument is "dummy" variable
  - used *only* to distinguish from pre-increment

Pre- and post-inc/decrement

Pre-decrement

- "decrement before returning value"
- `symbol` before variable
- return new value
- signature: `Type Type::operator -- () ;`

Post-decrement

- "decrement after returning value"
- `symbol` after variable
- return old value
- signature: `Type Type::operator -- (int) ;`
  - argument is "dummy" variable
  - used *only* to distinguish from pre-increment

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Recall Bézout's identity:

$$ax + by = \gcd(a, b)$$

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This explains `inverse()`

Finding powers

Recursive algorithm for a^k

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- if $k = 2\ell + 1$,

$$a^k = a^{2\ell+1} = a \times \left(a^\ell\right)^2$$

  so find a^ℓ, multiply to self, then to a

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pp. 174–175 #9.1, 9.5

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Hint on 9.1: From Bézout, find $c, d \in \mathbb{Z}$ such that

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$$0 + dn \equiv 1 \pmod{m} \quad \text{and} \quad cm + 0 \equiv 1 \pmod{n} \, .$$

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Notice this simplifies to

$$0 + dn \equiv 1 \pmod{m} \quad \text{and} \quad cm + 0 \equiv 1 \pmod{n} \, .$$

Let $x = adn + bcm$; then

$$\begin{aligned} x &\equiv adn + bcm \equiv a\,(dn) + 0 \equiv a \times 1 \equiv a \pmod{m} \\ y &\equiv adn + bcm \equiv 0 + b\,(cm) \equiv b \times 1 \equiv b \pmod{n} \, . \end{aligned}$$

So x is our solution, and the solution is unique modulo mn.

Summary

- Math stuff
  - additional use for Bézout
  - fast exponentiation
- Programming stuff
  - assignment can be construction
  - `constexpr`
  - construct before computing
  - inlining code
  - returning from arithmetic operations
  - `static` (variables *and* methods)
  - throwing exceptions