NORM TO/FROM METRIC

Definition 1 (Metric)**.** Let S be a set. A function $d : S \times S \to \mathbb{R}$ is a **metric** if it satisfies the following four criteria.

- $d(x,y) \geq 0$ for all $x,y \in V$
- $d(x,y) = 0$ iff $x = y$
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(y,z)$

Definition 2 (Norm)**.** Let V be a vector space over a field $\mathbb{R}$. A function $w : V \to \mathbb{R}$ is a **norm** if it satisfies the following four criteria.

- $w(v) \geq 0$ for all $v \in V$
- $w(v) = 0$ iff v is the zero vector
- $w(\alpha v) = |\alpha|\, w(v)$ for any $v \in V$ and any $\alpha \in \mathbb{F}$
- $w(u+v) \leq w(u) + w(v)$ for all $u,v \in V$

Fact 3. *A norm can be turned into a metric, via $d(x,y) = w(x-y)$. This is called the **induced metric**.*

Proof. Suppose w is a norm; we show the "induced metric" is indeed a metric. We concentrate on the triangle inequality, as the other properties should be easy. Let $x,y,z \in V$; then

$$\begin{aligned} d(x,z) = w(x-z) &= w(x+(-z)) \\ &= w((x-y)+(y-z)) \\ &\leq w(x-y) + w(y-z) \\ &= w(x-y) + |-1|\, w(y-z) \\ &= d(x,y) + d(y,z). \end{aligned}$$

□

Fact 4. *If a metric over a vector space satisfies the properties*

- $d(w,v) = d(w+u, u+v)$ *and*
- $d(\alpha u, \alpha v) = |\alpha|\, d(u,v)$

*then it can be turned into a norm, via $w(x) = d(x,0)$. This is called the **induced norm**.*

Proof. Suppose d is a metric that satisfies the additional properties; we show the "induced norm" is indeed a norm. We concentrate on the triangle inequality, as the other properties should be easy. Let $x,y,z \in V$; then

$$\begin{aligned} w(u+v) = d(u+v,0) &\leq d(u+v,(u+v)-v) + d((u+v)-v,0) \\ &= d(u+v,u) + d(u,0) \\ &= d(v+u,0+u) + d(u,0) \\ &= d(v,0) + d(u,0) \\ &= w(v) + w(u). \end{aligned}$$

□

Definition 5 (Hamming distance)**.** The **Hamming distance** between two vectors $u, v \in \mathbb{F}_q^n$ is the number of entries in which they differ.

Remark. If $q = 2$, and we treat the absolute value in $\mathbb{F}_2$ in the normal way, then the Hamming distance (our metric) satisfies the additional properties of Fact 4, since

- w and v differ in position i iff $w_i \neq v_i$ iff $w_i + u_i \neq u_i + v_i$ iff $w + u$ and $u + v$ differ in position i, and
- the only possible α in $\mathbb{F}_2$ are 0 or 1, so $d(\alpha u, \alpha v) = |\alpha| d(u, v)$ rather easily:
  - for $\alpha = 0$, we have $d(0 \cdot u, 0 \cdot v) = d(0, 0) = 0 = |0| d(u, v)$, while
  - for $\alpha = 1$, we have $d(1 \cdot u, 1 \cdot v) = d(u, v) = |1| d(u, v)$.

The Hamming metric can*not* induce a norm in $\mathbb{F}_q$ where $q \neq 2$, since we'd violate the second criterion. For instance, in $\mathbb{F}_3$ we have

$$d(2 \cdot 10, 2 \cdot 00) = d(20, 00) = 1 \neq 2 = 2 \cdot 1 = 2 \cdot d(10, 00).$$

That said, defining a norm in a finite field is always... *interesting*. We leave it at that.