MAT 681 ASSIGNMENT 2

Solve the following problems. Due in class Thursday, 8th September 2016.

1. Define $\pi_{\leq i}$ as the map from $\mathbb{X}$ to itself that "projects" a monomial in n variables to a monomial in i variables. For example,
$$\pi_{\leq 3}\left(x_1^5 x_2^4 x_4 x_5^2\right) = x_1^5 x_2^4 \,.$$
You can think of $\pi_{\leq i}$ as "chopping" variables x_{i+1}, x_{i+2}, ..., x_n off the monomial. More formally, if $1 \leq i \leq n$, then
$$\pi_{\leq i} : \mathbb{X} \to \mathbb{X} \quad \text{by} \quad \pi_{\leq i}\left(x_1^{a_1} \cdots x_n^{a_n}\right) = x_1^{a_1} \cdots x_i^{a_i}.$$
Show that the definition of the grevlex ordering is equivalent to the following:

Definition (Alternate definition of grevlex)**.** We say that $t <_\pi u$ if there exists $i \in \{1, 2, \ldots, m\}$ such that $\operatorname{tdeg}\left(\pi_{\leq k}(t)\right) = \operatorname{tdeg}\left(\pi_{\leq k}(u)\right)$ for each $k \in \{i+1, i+2, \ldots, n\}$ but $\operatorname{tdeg}\left(\pi_{\leq i}(t)\right) < \operatorname{tdeg}\left(\pi_{\leq i}(u)\right)$.

2. Let $\mathbf{w}$ be a vector in $(\mathbb{N}^+)^n$. A **weighted degree ordering with respect to w**, which we will denote $<_{\mathbf{w}}$, orders $t <_{\mathbf{w}} u$ whenever
$$\sum_{i=1}^{n} w_i \deg_{x_i} t \quad <_{\mathbf{w}} \quad \sum_{i=1}^{n} w_i \deg_{x_i} u \quad .$$
When $\mathbf{w}$ is clear from context, which it often is, we simply write, "a weighted ordering." For instance, if $\mathbf{w} = (1, 2, 3)$, then
$$x <_{\mathbf{w}} y <_{\mathbf{w}} z <_{\mathbf{w}} x^4 <_{\mathbf{w}} yz \quad .$$
   (a) Show that $<_{\mathbf{w}}$ is by itself insufficient for an admissible ordering. A counterexample using $\mathbf{w}$ in the example above will work, but for bonus points you should show it for arbitrary $\mathbf{w}$, since it gives you excellent practice refreshing your familiarity with vectors and linear algebra.
   (b) Find a value of $\mathbf{w}$ for which $<_{\mathbf{w}}$ is really the first test in the grevlex ordering.
   (c) For the weighted ordering with respect to $\mathbf{w} = (1, 3)$, sketch monomial diagrams that show:
      (i) the region of monomials less than or equal to $x^2 y^3$, and
      (ii) each line of monomials with an equal grading, up to total degree 9. (That is, you need not have any monomial with a grading larger than that of x^9 in your diagram.)
   (d) Propose a way to break ties in a weighted ordering, in a way that results in an admissible ordering. (*Hint:* Nothing stops you from adapting another ordering's approach to breaking ties. There is, therefore, more than one way to do this, but try to make it as uncomplicated as possible.)
   (e) Show that the ordering you describe in (d) is indeed admissible.

3. From *Peering into Advanced Mathematics*, all the exercises on pp. 23–24.