MAT 681 Assignment 1

August 25, 2016

Solve the following problems. Due in class Thursday, 1st September 2016.

1. Let $f = x + 4y + z$, $g = x + 8y + z$, and $h = x + 11y + 2z$.
   (a) What property of the system $\{f, g, h\}$ guarantees that it always has at least one solution, regardless of the underlying field? *Hint:* We didn't discuss it in class, but it's in the notes.
   (b) Find a triangular form for $\{f, g, h\}$, assuming the ground field is $\mathbb{Q}$. Is the number of solutions finite or infinite?
   (c) Find a triangular form for $\{f, g, h\}$, assuming the ground field is $\mathbb{Z}_2$. Is the number of solutions finite or infinite?
   (d) Find explicit solutions for the result of (b), parametrizing if there are infinitely many.
   (e) Find explicit solutions for the result of (c), parametrizing if there are infinitely many.
   (f) Why are the answers for (b) and (c) different?

2. Let $f = x + 4y + z$ and $g = y + 2z$.
   (a) By substituting for f and g into (1), show that

   $$yf - xg + 2zf - (4y + z)g = 0. \tag{1}$$

   (b) Starting with the left-hand side of (1), collect terms that have f or g in common, and factor those groups to an expression $pf + qg = 0$. What's special about p and q?

   By the way, you're looking at a **syzygy** of f and g: a pair (p, q) such that $pf + qg = 0$.

   (c) Notice that $\{f, g\}$ constitutes a linear system in triangular form. Now consider an *arbitrary* pair of linear polynomials f and g from a system in triangular form; that is, $f = t + f'$, $g = u + g'$, $t \neq u$, and all the monomials are linear. Generalize the syzygy hidden in (1), which you exposed in part (b), to find an equation similar to that in (1).
   (d) In class I showed that the ideal generated by $x^2 + y^3 = 4$ and $xy = 1$ also contains the polynomial $x + y^4 - 4y$, whose "leading variable" appears to be x. This was surprising because x is a multiple of neither x^2, y^3, nor xy. Use (c)'s result to explain why canceling leading variables of *general, linear* polynomials f and g in triangular form can never produce polynomials in the ideal generated by f and g whose leading variables are not multiples of the leading variables of f and g — *even if we allow multiplication by monomials!*