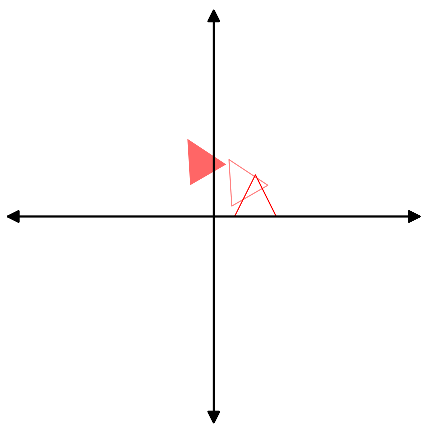
# A better explanation of the change of coordinates

September 21, 2013

Suppose we have an isometry $f = t_{(-2,1)}\rho_{\pi/6}$, in the standard coordinate system. In matrix notation,

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

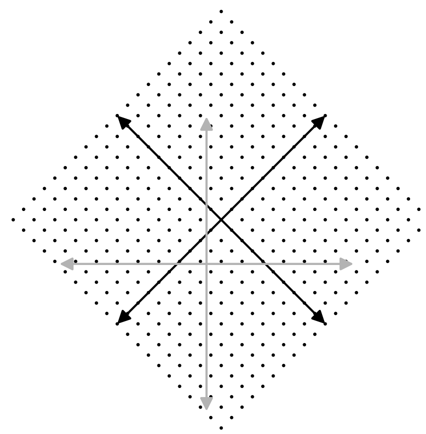
We illustrate this bellow with a triangle, that is first rotated off the $x$-axis, then shifted left 2 and up 1.



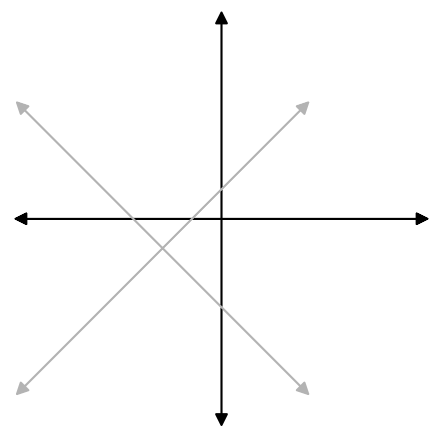
Now suppose we change our coordinates, according to the isometry $\eta = t_{(1,3)}\rho_{\pi/4}$. In matrix notation,

$$\eta\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

We illustrate this below: the original coordinate system is in gray, while the new coordinate system is in black. Each dot represents a point of the lattice $\mathbb{Z} \times \mathbb{Z}$.



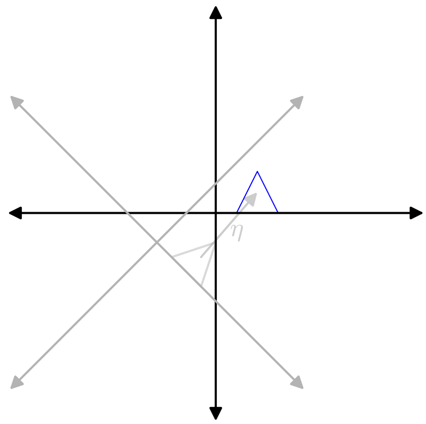
Let's adjust our point of view so that the new axes (black) look "normal", while the old axes (gray) look askance.



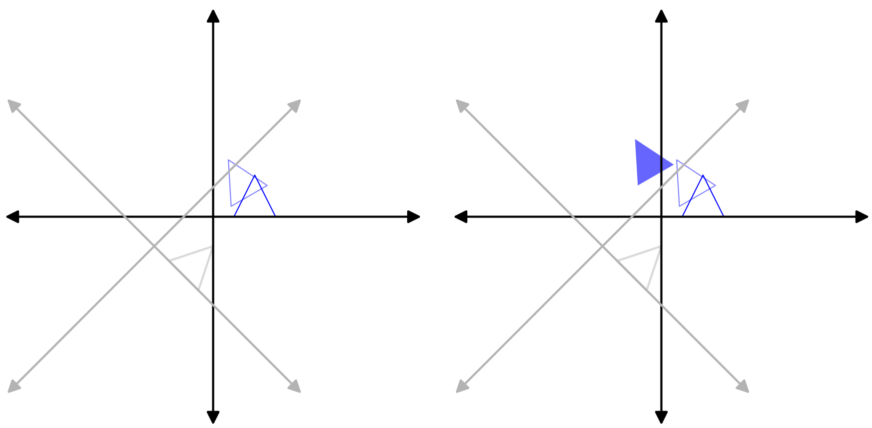
As explained in class, the isometry $\eta^{-1} f \eta$ is equivalent in this new coordinate system to the isometry $f$ in the old coordinate system. That means it should rotate the triangle *from* the same original position *to* the same final position *even though the coordinate system is different*. We can see algebraically how it does this by:

- applying $\eta$ to transform the triangle into the new coordinate system;
- applying $f$ *in the new coordinate system;* and
- applying $\eta^{-1}$ to transform the triangle back to the old coordinate system.

We apply $\eta$ first. It moves the original triangle into a triangle (blue) that has an equivalent position in the new coordinate system to its position (gray) in the old coordinate system.



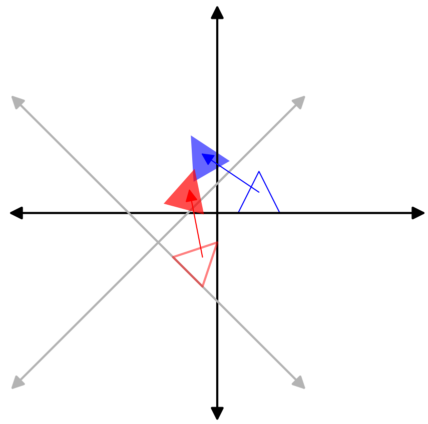

Now we apply $f$ in the new coordinate system: first a rotation, then a translation. When we are done, we have applied $f\eta$.



Finally, we apply $\eta^{-1}$. In matrix notation,

$$\eta^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{pmatrix} \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \end{pmatrix}\right].$$

(Notice that we apply the inverse translation *before* the inverse rotation.) In the figure below, we see this by the blue, filled triangle moving to the red, filled triangle in the old coordinate system. The arrows show that the end result successfully applied $f$ in the new coordinate system in a way that was equivalent to $f$ in the new coordinate system.



**Exercises.** The vertices of the red triangle in the original coordinate system were $(1, 0)$, $(3, 0)$, and $(2, 2)$.

(a) Compute the vertices of the red triangle in the new coordinate system.
*Hint:* Think about the fourth illustration. Or, think about the next problem.

(b) Applying $\eta$ to the vertices of the red triangle in the new coordinate system gives us the vertices of the first blue triangle in the new coordinate system. Show that these are the same as the vertices of the red triangle in the *original* coordinate system.

(c) Compute the vertices of the blue, filled triangle in the new coordinate system.

(d) Compute the vertices of the red, filled triangle in the new coordinate system.

(e) Explain how you already know the vertices of the red, filled triangle in the old coordinate system.