Modern Algebra 1 Section 1 · Assignment 5

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Exercise 1. *(pg. 52 Warm Up a) Compute the sum, difference, and product of the polynomials*

$$1-2x+x^3-\frac{2}{3}x^4 \quad \textit{and} \quad 2+2x^2-\frac{3}{2}x^3$$

in $\mathbb{Q}[x]$.

Solution:
Sum: $3-2x+2x^2-(1/2)x^2-(2/3)x^4$
Difference: $-1-2x-2x^2+(5/2)x^3-(2/3)x^4$
Product:

$$\begin{aligned}\left(2+2x^2-(3/2)x^3\right)+\left(-4x-4x^3+3x^4\right)\\ +\left(2x^3+2x^5-(3/2)x^6\right)+\left(-(4/3)x^4-(4/3)x^6+x^7\right)=2-4x+2x^2\\ -(7/2)x^3+(8/3)x^4\\ +2x^5-(17/6)x^6+x^7\end{aligned}$$

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Exercise 2. *(pg. 52 Warm Up b) Give the quotient and remainder when the polynomial* $2+4x-x^3+3x^4$ *is divided by* $2x+1$.

Solution:
The quotient is $(3/2)x^3-(5/4)x^2+(5/8)x-(27/16)$; the remainder is $5/16$. ◊

Exercise 3. *(pg. 53 Warm Up c) Give two polynomials* f *and* g*, where the degree of* $f+g$ *is strictly less than either the degree of* f *or the degree of* g.

Solution:
Answers may vary. What matters is that $\deg f=\deg g$ and the leading coefficients have the opposite sign; for example, $f=1$ and $g=-1$. ◊

Exercise 4. *(pg. 53 Warm Up d) Use the Root Theorem 4.3 to answer the following for polynomials in* $\mathbb{Q}[x]$. *Does* $x-2$ *divide* $x^5-4x^4-4x^3-x^2+4$*? Does* $x+1$ *divide* $x^6+2x^5+x^4-x^3+x$*? Does* $x+5$ *divide* $2x^3+10x^2-2x-10$*? Does* $2x-1$ *divide* $x^5+2x^4-3x^2+1$*?*

Solution:
Yes, yes, yes, no. (I used synthetic division with $x=2,-1,-5,1/2$. ◊

Exercise 5. *(pg. 53 Exercise 3) By Corollary 4.4 we know that a third-degree polynomial in* $\mathbb{Q}[x]$ *has at most three roots. Give four examples of third-degree polynomials in* $\mathbb{Q}[x]$ *that have 0, 1, 2, and 3 roots, respectively; justify your assertions. (Recall that here a root must be a rational number!)*

Solution:

For 0 roots, take x^3+2. This has no rational roots, only real and imaginary ones.

For 1 root, take $(x^2+2)(x+1)$. This has the rational root 1, and two irrational roots.

For 2 roots, take $(x-1)(x-2)^2$. This has the rational roots 1 and 2.

For 3 roots, take $(x-1)(x-2)(x-3)$. This has the rational roots 1, 2, and 3. ◊

Exercise 6. *(pg. 53 Exercise 5) Let n be an odd integer and consider the polynomial*

$$\Phi_n = x^n + x^{n-1} + \cdots + x + 1.$$

Use the Root Theorem 4.3 to argue that Φ_n has a linear factor. We call Φ_n a **cyclotomic** *polynomial; see Exercise 5.17 for more information.*

Solution:

Since n is odd, $n = 2k+1$ for some $k \in \mathbb{Z}$. So Φ_n has $2k+2$ terms $x^n, x^{n-1}, \ldots, x^1, x^0$. This is $2(k+1)$ terms, which gives us an even number of terms.

Since Φ_n has an even number of terms, we can pair consecutive terms into $k+1$ groups, like so:

$$\Phi_n = \left(x^n + x^{n-1}\right) + \left(x^{n-2} + x^{n-3}\right) + \cdots + \left(x^1 + x^0\right).$$

Each group is a pair of *consecutive* terms, so each group has one term of odd degree and one term of even degree. Recall that $(-1)^a = -1$ if a is odd, and $(-1)^a = 1$ if a is even. Thus

$$\begin{aligned}\Phi_n(-1) &= \left((-1)^n + (-1)^{n-1}\right) + \left((-1)^{n-2} + (-1)^{n-3}\right) + \cdots + \left((-1)^1 + (-1)^0\right)\\ &= (-1+1) + (-1+1) + \cdots + (-1+1)\\ &= 0 + 0 + \cdots + 0\\ &= 0.\end{aligned}$$

By the Root Theorem 4.3, $x+1$ is a linear factor of Φ_n. ◊

Exercise 7. *(pg. 54 Exercise 6) Suppose that $f \in \mathbb{Q}[x]$, $q \in \mathbb{Q}$, and $\deg f > 0$. Use the Root Theorem 4.3 to prove that the equation $f(x) = q$ has at most finitely many solutions.*

Solution:

We show that f has at most finitely many roots by induction on n. Let $n \in \mathbb{N}$ be arbitrary, but fixed.

For the inductive base, assume that $n = 1$. Then f has the form $mx + b$, for some $m, b \in \mathbb{Q}$, and $m \neq 0$. So

$$f(x) = q \iff mx + b = q \iff x = \frac{q-b}{m}.$$

We see that f has at most one solution.

Now assume that $n > 1$, and that g has at most finitely many roots, for every $g \in \mathbb{Q}[x]$ where $1 \leq \deg g < n = \deg f$. Assume that the equation $f(x) = q$ has at least one solution. Write $p = f - q$. By closure, $p \in \mathbb{Q}[x]$. Also,

$$p(x) = 0 \iff f(x) - q = 0 \iff f(x) = q,$$

so every $a \in \mathbb{Q}$ is a solution to $f(x) = q$ iff it is also a root of p. Recall that we assumed that $f(x) = q$ had at least one solution, and let a designate such a solution. So a is a root of p. By the Root Theorem 4.3, $x - a$ is a factor of p. Thus

$$p = q(x-a)$$

where $q \in \mathbb{Q}[x]$. By Theorem 4.1, $\deg q = \deg p - \deg(x-a) = n-1$. Since $\deg q < n$, the inductive hypothesis implies that $q(x) = 0$ has at most finitely many solutions. So q has finitely many roots. Since q has finitely many roots, and p has exactly one more root, p has finitely many roots. Recall that roots of p are solutions of the equation $f(x) = q$. Since p has finitely many roots, the equation $f(x) = q$ has at most finitely many solutions. ◊

Exercise 8. *(pg. 54 Exercise 13) Find a non-zero polynomial in $\mathbb{Z}_4[x]$ for which $f(a) = 0$, for all $a \in \mathbb{Z}_4$.*

Solution:

If $f(a) = 0$ for all $a \in \mathbb{Z}_4$, then every element $a \in \mathbb{Z}_4$ is a root of f. By the Root Theorem 4.3, $x - a$ is a factor of f for every $a \in \mathbb{Z}_4$. It is useful that there are only four elements of $\mathbb{Z}_4$: $[0],[1],[2],[3]$. Thus the solution is

$$f = x(x-[1])(x-[2])(x-[3]).$$

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