Modern Algebra 1 Section 1 · Assignment 1

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Exercise 1. *(pg. 11 Warm-up c) Suppose we have an infinite row of dominoes, set up on end. What sort of induction argument would convince us that knocking down the first domino will knock them all down?*

Solution:

We have to show that if you knock down any one domino, then it knocks down the one behind it. ◊

Exercise 2. *(pg. 11 Exercise 2) Prove using mathematical induction that for all positive integers n,*

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(2n+1)(n+1)}{6}. \tag{1}$$

Solution:

We proceed by induction.

For the inductive base, assume $n = 1$.

The left hand side of (1) simplifes to

$$1^2 = 1$$

and the right hand side simplifies to

$$\frac{1(2 \cdot 1 + 1)(1+1)}{6} = \frac{1 \cdot 3 \cdot 2}{6} = 1.$$

Since the two sides are equal, (1) is true for $n = 1$.

Now assume $n > 1$, and assume that $1^2 + 2^2 + \cdots + k^2 = k(2k+1)(k+1)/6$ for all $k : 1 \leq k < n$.

On the left hand side of (1), we get

$$1^2 + 2^2 + \cdots + n^2 = \left(1^2 + 2^2 + \cdots + (n-1)^2\right) + n^2.$$

By the inductive hypothesis, this simplifies as

$$\begin{aligned}\left(1^2 + 2^2 + \cdots + (n-1)^2\right) + n^2 &= \frac{(n-1)(2(n-1)+1)((n-1)+1)}{6} + n^2 \\ &= \frac{(n-1)(2n-1) \cdot n}{6} + \frac{6n^2}{6} \\ &= \frac{2n^3 + 3n^2 + n}{6}.\end{aligned}$$

On the right hand side, we get

$$\frac{n(2\cdot n+1)(n+1)}{6}=\frac{\left(2n^2+n\right)(n+1)}{6}$$
$$=\frac{2n^3+3n^2+n}{6}.$$

We see that the left and right hand sides of (1) are equal.
Hence $1^2+2^2+\cdots+n^2=n(2n+1)(n+1)/6$. ◊

Exercise 3. *(pg. 11 Exercise 3) You probably recall from your previous mathematical work the* triangle inequality: *for any real numbers x and y,*

(2) $$|x+y|\le|x|+|y|.$$

Accept this as given (or see a calculus text to recall how it is proved). Generalize the triangle inequality, by proving that

(3) $$|x_1+x_2+\cdots+x_n|\le|x_1|+|x_2|+\cdots+|x_n|,$$

for any positive integer n.

Solution:

We proceed by induction.
For the inductive base, assume $n=1$.
Let $x_1\in\mathbb{R}$ be arbitrary, but fixed.
It is obvious that $|x_1|\le|x_1|$.
We have shown that the assertion is true for $n=1$.
Assume $n>1$ and $|x_1+x_2+\cdots+x_k|\le|x_1|+|x_2|+\cdots+|x_k|$ for all $k:1\le k<n$.
Let $x_1,x_2,\ldots,x_n\in\mathbb{R}$ be arbitrary, but fixed.
The left hand side of (3) is

$$|x_1+x_2+\cdots+x_n|\le\left|(x_1+x_2+\cdots+x_{n-1})+x_n\right|.$$

This is a sum of two integers. By the basic triangle inequality (2),

(4) $$\left|(x_1+x_2+\cdots+x_{n-1})+x_n\right|\le\left|x_1+x_2+\cdots+x_{n-1}\right|+|x_n|.$$

By the inductive hypothesis,

$$\left|x_1+x_2+\cdots+x_{n-1}\right|\le|x_1|+|x_2|+\cdots+\left|x_{n-1}\right|.$$

Substituting this fact into (4), we obtain (3). ◊

Exercise 4. *(pg. 12 Exercise 4) Given a positive integer n, recall that $n!=1\cdot2\cdot3\cdots n$ (this is read as* n factorial*). Provide an inductive definition for $n!$. (It is customary to actually start this definition at $n=0$, setting $0!=1$.)*

Solution:

Define $0!=1$ and $n!=(n-1)!\times n$. ◊

Exercise 5. *(pg. 12 Exercise 5) Prove that $2^n<n!$ for all $n\ge4$.*

Solution:

We proceed by induction.
For the inductive base, assume $n=4$.

Then $2^4 = 16 < 24 = 4!$.

Assume $n > 4$, and that $2^k < k!$ for all $k : 4 \leq k < n$.

We have

$$n! = (n-1)! \times n > 2^{n-1} \times n > 2^{n-1} \times 2 = 2^n.$$

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Exercise 6. *(pg. 13 Exercise 14) In this problem you will prove some results about the* ***binomial coefficients****, using induction. Recall that*

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

where n is a positive integer, and $0 \leq k \leq n$.

(a) Prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \tag{5}$$

for $n \geq 2$ and $k < n$. Hint: *You do not need induction to prove this. Bear in mind that $0! = 1$.*

(b) Verify that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$. Use these facts, together with part a, to prove by induction on n that $\binom{n}{k}$ is an integer, for all k with $0 \leq k \leq n$.

(c) Use part a and induction to prove the ***Binomial Theorem:*** *For non-negative n and variables x, y,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Solution:

(a) We start with the right hand side of (5):

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!}.$$

Get a common denominator; then

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)! \cdot (n-k)}{k!(n-k)!} + \frac{(n-1)! \cdot k}{k!(n-k)!} \\
&= \frac{(n-1)! \cdot (n-k) + (n-1)! \cdot k}{k!(n-k)!} \\
&= \frac{(n-1)! \cdot ((n-k)+k)}{k!(n-k)!} \\
&= \frac{(n-1)! \cdot n}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

(b) First, by definition

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1 \in \mathbb{Z} \tag{6}$$

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{1}{0!} = 1 \in \mathbb{Z}. \tag{7}$$

Let $k \in \mathbb{Z}$ be arbitrary, but fixed. We prove that $\binom{n}{k}$ is an integer by induction on n.

For the inductive base, assume $n = 0$.

If $k = 0$, equation (6) shows that $\binom{0}{k}$ is an integer.

By definition, $0 \le k \le n$, so there is nothing more to show for $n = 0$.

Assume $0 < n$, and assume that $\binom{i}{k}$ is an integer for all $i, k : 0 \le k \le i < n$.

We know from part (a) that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Since $n - 1 < n$, the inductive hypothesis implies that $\binom{n-1}{k}$ and $\binom{n-1}{k-1}$ are both integers. By closure, the sum of two integers is also an integer. Thus $\binom{n}{k}$ is an integer.

(c) We proceed by induction on n.

For the inductive base, assume $n = 0$. Then

$$(x+y)^0 = 1$$

and

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{0}{0} x^0 y^0 = 1.$$

The two sides are equal.

Assume $n > 0$, and assume

$$(x+y)^k = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^i$$

for all $k : 0 \le k < n$.

By the inductive hypothesis,

$$\begin{aligned}(x+y)^n = (x+y)(x+y)^{n-1} &= (x+y)\sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-1-k}y^k\\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k}y^k + \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-1-k}y^{k+1}\\
&= \binom{n-1}{0}x^n + \sum_{k=1}^{n-1}\binom{n-1}{k}x^{n-k}y^k\\
&\quad + \sum_{k=0}^{n-2}\binom{n-1}{k}x^{n-1-k}y^{k+1} + \binom{n-1}{n-1}y^n.\end{aligned}$$

Reindex the second summation, so that

$$\begin{aligned}(x+y)^n = \binom{n-1}{0}x^n &+ \sum_{k=1}^{n-1}\binom{n-1}{k}x^{n-k}y^k\\
&+ \sum_{k=1}^{n-1}\binom{n-1}{k-1}x^{n-k}y^k + \binom{n-1}{n-1}y^n.\end{aligned}$$

(What happened here? I added 1 to the k's that index the sum, which requires me to subtract 1 from the k's in the formula. This is called reindexing, and is a useful tool. You can verify that it is correct by writing out a few terms of the sum before and after reindexing. **I do not expect anyone to have written the proof this way.** In fact I showed students a different way of writing the same thing myself, but you need to see it sooner or later, so there it is!)

Combining like terms, we have

$$\begin{aligned}(x+y)^n &= \binom{n-1}{0}x^n + \sum_{k=1}^{n-1}\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)x^{n-k}y^k + \binom{n-1}{n-1}y^n\\
&= \binom{n-1}{0}x^n + \sum_{k=1}^{n-1}\binom{n}{k}x^{n-k}y^k + \binom{n-1}{n-1}y^n\\
&= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.\end{aligned}$$

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