PROOFS TO PRESENT (ROUND 3)

MAT 423

1. Let M be a monoid. A **monoid (left) ideal** of M is a set A such that $A \subseteq M$ and $ma \in A$ for any $m \in M$ and for any $a \in A$. We say that a monoid ideal A is **generated** by a set $B \subseteq A$ if for every $a \in A$ we can find $m \in M$ and $b \in B$ such that $a = mb$.
   (a) Let $N = \{0, 2, 4, \ldots\}$. Show that N is *not* a monoid ideal of $\mathbb{N}$.
   (b) Let $N = \{3, 4, 5, \ldots\}$. Show that N *is* a monoid ideal of $\mathbb{N}$.
   (c) Show that any monoid ideal of $\mathbb{N}$ has the form $\{a, a+1, a+2, \ldots\}$ $\exists a \in \mathbb{N}$. (Hint: WOP.)
   (d) Explain why this means every monoid ideal of $\mathbb{N}$ has exactly one generator.
2. Let $(G, \times)$, $(H, \otimes)$, and $(K, *)$ be groups.
   (a) Show that the identity funciton $I(g) = g$ is an isomorphism on G.
   (b) Suppose we know $G \cong H$. That means there is an isomorphism $f : G \to H$. Every isomorphism is one-to-one and onto. *That* means f has an inverse function $f^{-1} : H \to G$, also one-to-one and onto. Show that f^{-1} is also a homomorphism, so that $H \cong G$.
   Hint: You need to show that $f^{-1}(xy) = f^{-1}(x) f^{-1}(y)$ for every $x, y \in G$. You already know f is an isomorphism, so you can find $a, b \in G$ such that $f(a) = x$ and $f(b) = y$. Use these facts, along with the fact that f is an isomorphism, to finish the job.
   (c) Suppose we know $G \cong H$ and $H \cong K$. That means there exist isomorphisms $\mu : G \to H$ and $\varphi : H \to K$. Let $\psi = \varphi \circ \mu$; that is, ψ is the composition of the functions g and h. Explain why $\psi : G \to K$, and show that ψ is also a homomorphism.
3. Define a relation $\bowtie$ on $\mathbb{Q}$, the set of rational numbers, in the following way:
   $$a \bowtie b \quad \text{if and only if } a - b \in \mathbb{Z}.$$
   (a) Is $2/3 \bowtie 1/2$? Is $12/5 \bowtie -3/5$?
   (b) Show that $\bowtie$ is an equivalence relation: reflexive, symmetric, transitive.
   (c) How do we know that $\bowtie$ partitions $\mathbb{Q}$?
   (d) Show that $a \bowtie b$ if they have the same sign and the same fractional part. (The "fractional part" of a number is part that appears in the decimal expansion after the decimal point.)
4. This problem considers divisibility of ring elements as a relation. Recall that $a \mid b$ if and only if we can find $q \in R$ such that $aq = b$.
   (a) Show that divisibility is both reflexive and transitive.
   (b) Now show that divisibility is not symmetric.
   (c) So divisibility is not an equivalence relation. Can it be a partition?
5. These questions concern Lagrange's Theorem and its consequences.
   (a) Suppose a group has order 8, but is not cyclic. Why must g^4 be the identity for every g in the group?
   (b) Let g be a finite group, and $g \in G$. Why must $g^{|G|}$ be the identity?
   (c) Suppose a group G has prime order; that is, $|G| = p$ where p is prime. Show that G has no proper subgroup.