PROOFS TO PRESENT (ROUND 2)

MAT 423

1. Suppose G and H are groups, and there exists a homomorphism $f : G \to H$.
   (a) Show that if $f(g) = h$ and $\operatorname{ord}(g) < \infty$, then $\operatorname{ord}(h) \mid \operatorname{ord}(g)$.
   (b) Show that if G is cyclic, H is cyclic, and f is onto, then $|H|$ divides $|G|$.

2. Recall the set of orthogonal matrices, $O(n) \subsetneq \mathbb{R}^{n\times n}$. In Question 3·94 you showed that every orthogonal matrix had determinant ± 1. Let $SO(n)$ be the subset of $O(n)$ consisting of matrices with determinant 1.
   (a) Show that $SO(n)$ is a group.
   (b) Show that *any* $A \in SO(n)$ has the form

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

   for some $\alpha \in \mathbb{R}$. (*Hint:* Use the technique I used in class to obtain a generic form for elements of $O(n)$. One of the resulting equations should look like a circle. What is a "trigonometric" equation of the circle? Proceed from there.)

3. Let $m, n \in \mathbb{Z}\setminus\{0\}$. Recall that $\mathbb{Z}_m \times \mathbb{Z}_n$ is a group under addition, with identity $(0,0)$.
   (a) Show that $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not cyclic.
   (b) Are there any values of m, n such that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic?
   (c) Show that $\pi_1 : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_m$ by $\pi_1(a,b) = a$ and $\pi_2 : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_n$ by $\pi_2(a,b) = b$ are both homomorphisms. (We call π_1 and π_2 **projection homomorphisms**.)
   (d) Can π_2 be an isomorphism?

4. Let $m, n \in \mathbb{Z}\setminus\{0\}$. Recall that $\mathbb{Z}_m \times \mathbb{Z}_n$ is a group under addition, with identity $(0,0)$. This problem uses part of #3, so you may want to review its results, but you don't have to prove #3 to do #4.
   (a) Show that $\iota_1 : \mathbb{Z}_m \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $\iota_1(a) = (a,0)$ and $\iota_2 : \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $\iota_2(b) = (0,b)$ are both homomorphisms. (We call ι_1 and ι_2 **injection homomorphisms**.)
   (b) A sequence of homomorphisms

$$G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3$$

   is **exact** if the image of f is the kernel of g. Show that if $G_1 = \mathbb{Z}_m$, $G_2 = \mathbb{Z}_m \times \mathbb{Z}_n$, $G_3 = \mathbb{Z}_n$, $f = \iota_1$, and $g = \pi_2$ (where ι_1 and π_2 are the injection and projection homomorphisms, respectively) then the sequence is exact.

5. Suppose G and H are groups, and there exists a homomorphism $f : G \to H$ such that f is onto.
   (a) Show that if G is cyclic, then so is H.
   (b) Recall that $\mathbb{Z}$ is cyclic, and let $d \in \mathbb{N}\setminus\{0\}$. Suppose you didn't already know $\mathbb{Z}_d$ were cyclic; why would (a) show you that it is?
   (c) Show the converse of (a) is false; that is, H can be cyclic even if G is not. (*Hint:* #3 helps.)