PROOFS TO PRESENT

MAT 423

1. Suppose R is a ring, and $f : R \to \mathbb{Z}$ is a one-to-one homomorphism. Show that R is an integral domain.

2. Let $f : \mathbb{Z}_6 \to \mathbb{Z}_2 \times \mathbb{Z}_3$ by the rule $f(a) = ([a]_2, [a]_3)$. For instance, $f(4) = ([4]_2, [4]_3) = (0, 1)$.
   (a) Compute the image of $\mathbb{Z}_6$.
   (b) How do you know f is one-to-one and onto?
   (c) Show that f is a homomorphism.
   (d) Show that f is an isomorphism; that is, $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.
   (e) Show that it is *not* always true that $\mathbb{Z}_{m \times n} \cong \mathbb{Z}_m \times \mathbb{Z}_n$. Why is $\mathbb{Z}_6$ special?

3. Let $R = \mathbb{Q}[x]$, the ring of integer polynomials, and $f(x) = x^2 - 2$.
   (a) Explain why f has no root in $\mathbb{Z}$.
   (b) Let P be the set of remainders of elements of R after division by f.
      (i) What form do the elements of P have?
      (ii) Simplify x^2, $(x+1)(x-1)$, $(x+2)(x-2)$, x^4 *as elements of* P.
      (iii) Show that x is a root of $y^2 - 2 \in P[y]$.
      (iv) Show that P is a ring.
      (v) Show that P is a field.

4. Let $R = \mathbb{Z}_2[x]$, the ring of polynomials whose coefficients are integers, modulo 2, and $f(x) = x^2 + x + 1$.
   (a) Explain why f has no root in $\mathbb{Z}_2$.
   (b) Let P be the set of remainders of elements of R after division by f.
      (i) What form do the elements of P have?
      (ii) Compute x^2, $x^2 + 1$, $x(x+1)$, x^3, $x\left(x^2+1\right)$ *as elements of* P.
      (iii) Show that x is a root of $y^2 + y + 1 \in P[y]$.
      (iv) Show that P is a ring.
      (v) Show that P is a field.

5. Let $R = \mathbb{Z}[x]$, the ring of integer polynomials, and $f(x) = x^4 - 4$.
   (a) Explain why f has no root in $\mathbb{Z}$.
   (b) Let P be the set of remainders of elements of R after division by f.
      (i) What form do the elements of P have?
      (ii) Simplify x^4, $\left(x^2+2\right)\left(x^2-2\right)$, $\left(x^2+2\right)\left(x^3-2\right)$, x^5 *as elements of* P.
      (iii) Show that x is a root of $y^4 - 4 \in P[y]$.
      (iv) Show that P is a ring.
      (v) Show that P is *not* an integral domain.
      (vi) Explain why P cannot be a field.