Question 1.17 .
If a set S has m elements and a set T has n elements, how many elements will $S \times T$ have? Explain why.

If $S = T$, we can write S^2 instead of $S \times T$. Hence we can abbreviate the lattice of Ideal Nim as $\mathbb{N}^2$.

When needed, we can chain sets in the Cartesian product to make sequences longer than mere pairs; we can even describe all infinite sequences of integers as

$$\mathbb{Z}^\infty = \prod_{i=1}^{\infty} \mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots = \{(a, b, c, \dots) : a, b, c, \dots \in \mathbb{Z}\}.$$

That new symbol, $\prod$, means "product", much as Σ means "sum". Writing phrases like "the first element of P" or "the four hundred twenty-fifth element of P" all the time grows cumbersome, so we'll adopt the convention that if P is a sequence of numbers, then p_i will stand for the ith element of P. For example, if $P = (5, 8, 3, -2)$ then $p_1 = 5$ and $p_4 = -2$.

Relations and the size of a set

When we want to prove that a set's elements satisfies a certain property, it's relatively easy to do when the set has only a few elements; with a **brute force** approach, we check every element of the set. This approach doesn't work well if the set is rather large, and is absolutely impossible when the set's size is infinite.

So we need some sort of tool to work with large and infinite sets.

Definition 1.18. A **relation** between two sets S and T is a subset of $S \times T$. We call S the **domain** and T the **range**. For instance, the pairings of fingers is a relation on $F \times F$, where the set of fingers, while $S \times T$ is itself a relation. A **function** is any relation $F \subsetneq S \times T$ such that every $s \in S$ corresponds to exactly one $(s, t) \in F$. Put another way, any two (a, b) and (c, d) in F satisfy $a \neq c$ (but $b = d$ is okay). If F is a function, we write $F : S \to T$ instead of $F \subseteq S \times T$, and $F(s) = t$ instead of $(s, t) \in F$. In this latter case, we call s the **input** and t the **output**.

Two special kinds of functions are worth mentioning. The first is the kind where every input has a unique output. Stated more precisely, a function $f : S \to T$ is **one-to-one** if and only $f(r) = f(s)$ implies that $r = s$. (Here, r and s are obviously in S.) A somewhat informal way of saying the same this is that if two outputs are the same, the inputs must also have been the same.

To describe the other kind of function, we need to introduce another term. The **image** of a function, written $\text{Img}(f)$, is the subset of the range the function actually maps to; that is, if $f : S \to T$, then $\text{Img}(f) = \{t \in T : \exists s \in S\, f(s) = t\}$. A function is **onto** if its image equals its range; that is, $\text{Img}(f) = T$, or, for any $t \in T$ we can find $s \in S$ such that $f(s) = t$.

If a function is both one-to-one and onto, we refer to it as a **bijection**.

Example 1.19. Suppose $S = \mathbb{Z}$ and $T = 2\mathbb{Z}$. Let $f : S \to T$ by $f(s) = 2s$. We claim f is a bijection.

First we have to show it's a function: that is, any input has only one output. This is fairly clear from its definition; multiplication gives only one result.

Next we have to show it's one-to-one. Suppose that we can find $r, s \in S$ such that $f(r) = f(s)$. By definition of f, we have $2r = 2s$. We can rewrite this as $2(r-s) = 0$. You should know from past experience that the product of two integers is zero only if one of the integers is zero (this is called the Zero Product Property, and you will see it again[12]); since $2 \neq 0$, we must have $r - s = 0$. From here it is evident that $r = s$. We chose r and s arbitrarily, so f is one-to-one.

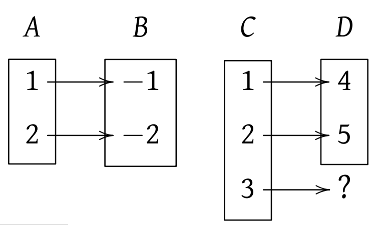
Finally, we have to show it's onto. Let $t \in T$. By definition of T, we can find $z \in \mathbb{Z}$ such that $t = 2z$. Observe that $z \in S$ and $f(z) = t$. We chose z arbitrarily, so f is onto.

One application of functions you probably haven't seen is to give a precise meaning to the concept of a set's size. Determining a set's size seems easy enough when the sets are finite; with the brute force approach, we just count the number of elements. If the sets are not finite, the question is actually not so easy as you might think; not all infinite sets have the same size.

Definition 1.20. Two sets S and T have the same size (or **cardinality**) if you can match each element of S to a unique element of T, covering all the elements of T in the process. More precisely, S and T have the same cardinality if you can create a mapping from S to T where

- each element of S maps to a unique element of T (so the function is one-to-one), and
- for any element of T, you can find an element of S that maps there (so the function is onto).

For example, the sets $A = \{1, 2\}$ and $B = \{-1, -2\}$ have the same cardinality because I can match them as follows, while the sets $C = \{1, 2, 3\}$ and $D = \{4, 5\}$ do not, because I cannot find a unique target for at least one element of C:



[12] You may wonder why we don't divide by 2. We could in this case, but there will be occasions where we have to solve similar equations and we *can't* necessarily go that route. The technique used here often works in those cases, so we want to introduce it here.