"TEAM PROJECT #2"

MAT 4/523 FALL 2011

1. Instructions

This assignment is due the beginning of class (*not* "midnight," "before I leave by office," or even "before the end of class") on Wednesday, the 16th of November.

You will do something a little different this time. Each exercise is worth a certain number of points, and each member will assume responsibility for one or more exercises.

- No exercise may go undone.
- Each team member must take responsibility for at least ten points.
- If one member's responsibility is n points, no member's responsibility can exceed $n+10$. (If this requirement is too hard to meet, talk with me.)
- Team members may *not* divide the points of an exercise among them; the exercise is entirely the responsibility of the student who takes it. The exception to this rule is Exercise 8; its three parts may be divided among more than one team member, if desired.
- Exercises with concrete examples would be useful to the students working on the theoretical problems. So, students working on the former should try to finish them quickly.

The team should present their work to each other at some point, and give each other advice on how to write the argument more clearly, correctly, etc. However, each team member will associate her or his name with the problems s/he takes responsibility for.

Half your grade is determined by the score you earn on the exercises of your responsibility. A quarter is determined by the team's score. The rest is determined by peer evaluation.

Gp 1	**Gp 2**	**Gp 3**	**Gp 4**
Nevada Brown	Chardae Cousin	Stradford Goins	Austin Andries
Eric Gustaffson	Taylor Kilman	Brenton Jones	Aaron Ayers
Kristopher Katterjohn	Cononiah Watson	Charles Schloemer	Patrick Lambert
			Kristie West

2. Background material

Notation. Throughout this assignment, G is a group, and $H < G$.

Definition 1. Let $g, x, y, z \in G$. The **commutator of** x **and** y is $[x,y] = x^{-1}y^{-1}xy$. The **conjugation** of g by z is $g^z = zgz^{-1}$.

Definition 2. The **commutator subgroup of** G is the intersection of all subgroups of G that contain $[x,y]$ for all $x, y \in G$. We denote this subgroup by both $[G,G]$ and G'.

Fact. *By Exercise 3.19, $[G,G] < G$. So, another way of defining the commutator subgroup, is as the smallest subgroup of G that contains $[x,y]$ for all $x, y \in G$.*

3. EXERCISES

1. **(10 pts)** Define the **conjugation of H by g** as
$$gHg^{-1} = \{h^g : h \in H\}.$$
Show that $H \triangleleft G$ if and only if $H = gHg^{-1}$ for all $g \in G$.

2. **(2 pts)** Compute $[K_4, K_4]$, where K_4 is the Klein 4-group.

3. **(3 pts)** Compute $[\mathbb{Z}_4, \mathbb{Z}_4]$, where $\mathbb{Z}_4$ is the clockwork group, modulo 4.

4. **(3 pts)** Compute $[D_3, D_3]$.

5. **(5 pts)** Compute $[Q_8, Q_8]$.

6. **(15 pts)** Show that $[G, G] \triangleleft G$; that is, G' is a *normal subgroup* of G.
Hint: The "hard" way is to show that for all $g \in G$, $gG' = G'g$. This requires you to show two sets are equal. Any element of G' has the form $[x, y]$ for some $x, y \in G$. At some point, you will have to show that $g[x, y] = [w, z]g$ for some $w, x \in G$. Try to construct w and z that satisfy the equation; this requires a not-so-obvious trick with cancellation.
An "easier" way uses the result of Exercise 3.63, showing that $gG'g^{-1} = G'$ for any $g \in G$. Exercise 2.34 should help you see why $gG'g^{-1} \subseteq G'$; to show the reverse direction, show why any $g' \in G'$ has the form $g^{-1}[x^g, y^g]g$ for any $g \in G$, so $gG'g^{-1} \supseteq G'$.

7. **(5 pts)** Show that if $H \subseteq G$, then $H' \subseteq G'$.

8. **(3 pts)** Show that Q_8 is solvable.

9. *[An adaptation of Exercise 3.102]* In the textbook *God Created the Integers...* the theoretical physicist Stephen Hawking reprints, with commentary, some of the greatest mathematical results in history. One excerpt is from Evariste Galois' *Memoirs* on the solvability of polynomials by radicals. Hawking sums it up this way.

> To be brief, Galois demonstrated that the general polynomial of degree n could be solved by radicals if and only if every subgroup N of the group of permutations S_n is a normal subgroup. Then he demonstrated that every subgroup of S_n is normal for all $n \leq 4$ but not for any $n > 5$.
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> —p. 105

Hawking's explanation is completely wrong, and a simple investigation of D_3 explains why.[1] It may seem odd to use D_3 as a counterexample to a statement about S_n, but you will learn in Section 5.2 that the group S_3 is really the same as D_3.[2]
(a) **(2 pts)** D_3 has six subgroups. List them all.

[1]Perhaps Hawking was trying to simplify what Galois actually showed, and went too far. (I've done much worse in my lifetime.) In fact, Galois showed that a polynomial of degree n could be solved by radicals if and only if a corresponding group, now called its **Galois group**, was a solvable group. He then showed that the Galois group of $x^5 + 2x + 5$ was not a solvable group.

[2]To resurrect a term we used with monoids, S_3 is isomorphic to D_3. We will talk about group isomorphisms in Chapter 4.

(b) **(3 pts)** It is known that the general polynomial of degree 3 can be solved by radicals. According to Hawking's assertion, what must be true about all the subgroups of D_3?

(c) **(2 pts)** Why is Hawking's explanation of Galois' result "obviously" wrong?

4. Peer Evaluation

Separately from the rest of the questions, please evaluate *each* of the peers in your group on the following criteria, using a scale of 1–5. A score of 1 indicates that you *strongly disagree* with the statement; a score of 5 means that you *strongly agree* with the statement. Each peer should be evaluated on a separate sheet of paper. Both your peer's name *and yours* should appear on each evaluation, so that I know that you have completed the assignment. Although your peers will know their *cumulative* scores, they will not learn how each team member scored them.

1. This individual participated fully in the scheduled meeting(s).

2. To the best of my knowledge, this individual made a genuine effort to understand the concepts necessary to complete the exercises for which s/he was responsible.

3. This individual presented solutions to the exercises for which s/he was responsible in a *timely* fashion.
   Note: These solutions need not have been correct; it's a question of timeliness.

4. This individual presented her/his answers to the exercises for which s/he was responsible in a *clear* fashion.
   Note: These solutions need not have been timely or correct; it's a question of clarity of explanation.

5. To the best of my knowledge, this individual did, in fact, have an understanding of the exercises for which s/he was responsible.