HELP WITH 37.4

The following lemmata would be extremely useful for 37.4. The book's author kindly sent me another approach, so if you're really interested in this problem but can't swing this approach, let me know & I'll send you his hint.

Notation. Write $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Lemma 1. $\omega^3 = 1$.

Proof. You do this! □

Lemma 2. *The polynomial $x^3 - a = 0$ has roots $\sqrt[3]{a}$, $\omega\sqrt[3]{a}$, and $\omega^2\sqrt[3]{a}$.*

Proof. You do this! □

Lemma 3. *The polynomial $x^2 - b = 0$ has roots $\pm\sqrt{b}$.*

Proof. You do this! □

Main result. *Let $u = \sqrt[3]{a}$ and $v = \sqrt{b}$. The expansion of the polynomial*

$$(x-(u+v))(x-(u-v))(x-(u\omega+v))(x-(u\omega-v))\left(x-\left(u\omega^2+v\right)\right)\left(x-\left(u\omega^2-v\right)\right)$$

has integer coefficients.

Proof. You do this! □

Corollary. *The integer polynomial _____ has $\sqrt{2} + \sqrt[3]{3}$ as a root. (Fill in the blank!)*

Proof. You do this! □

Remark. Notice how we construct the polynomial in the theorem: the root of each *linear* factor contains *every* combination of the roots of $x^3 - a$ and $x^2 - b$.