MAT 305: Mathematical Computing

Solving equations in Sage

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Outline

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Exact solutions

- Many equations can be solved without rounding
  - *exact solutions*
  - Solving by radicals: old, important problem
    - Niels Abel, Evariste Galois, Joseph Lagrange, Paolo Ruffini, …
  - Special methods
- Others require approximate solutions

The `solve()` command

`solve(`*eqs*`,` *vars*`)` where

- *eqs* is an equation or a list of equations
- *vars* is an indeterminate or list of indeterminates to solve for
    - unlisted indetermintes treated as constants
- returns a list of solutions *if* Sage can solve *eqs* exactly

$= \neq ==$

FACT OF PYTHON

- `=` (single)
  - assignment of a value to a symbol
  - `f = x^2 - 4` assigns the value $x^2 - 4$ to f
  - "let $f = x^2 - 4$"
- `==` (double)
  - two quantities are equal
  - `16==4^2` is *true*
  - `16==5^2` is *false*
  - `16==x^2` is *conditional*; it depends on the value of x
- Confuse the two? *naughty user*

Example

```
sage: 16==4^2
True

sage: 16==5^2
False

sage: 16==x^2
16 == x^2
```

(cannot simplify the expression)

Univariate polynomials

```
sage:  solve(3*x+1==4*(x-2)+3,x)
[x == 6]

sage:  solve(x^2==-1,x)
[x == -I, x == I]
```

(`I`* represents $\sqrt{-1}$)*

```
sage:  solve(x^5+2*x+1==0,x)
[0 == x^5 + 2*x + 1]
```

(Sage cannot find exact solution)

Unknown constants

```
sage: var('a b c')
(a, b, c)
sage: solve(a*x^2+b*x+c==0,x)
[x == -1/2*(b + sqrt(-4*a*c + b^2))/a,
 x == -1/2*(b - sqrt(-4*a*c + b^2))/a]
```

(quadratic formula!)

Copying solutions not always a good idea

```
sage: solve([3*x^3-4*x==7],x)
[x == -1/2*(1/54*sqrt(3713) + 7/6)^(1/3)*(I*sqrt(3)
+ 1) + 1/9*(2*I*sqrt(3) - 2)/(1/54*sqrt(3713) +
7/6)^(1/3), x == -1/2*(1/54*sqrt(3713) +
7/6)^(1/3)*(-I*sqrt(3) + 1) + 1/9*(-2*I*sqrt(3) -
2)/(1/54*sqrt(3713) + 7/6)^(1/3), x ==
(1/54*sqrt(3713) + 7/6)^(1/3) + 4/9/(1/54*sqrt(3713)
+ 7/6)^(1/3)]
```

ouch!

Assign, use [ ]

To extract values from solutions, assign and use `[ ]`

Example

```
sage: sols = solve([x^4-1==0],x)
sage: sols
[x == I, x == -1, x == -I, x == 1]
sage: sols[0]
x == I
sage: sols[1]
x == -1
sage: sols[3]
x == 1
```

But I want only the number...!

- Every equation has a right hand side
- Use `.rhs()` command
    - "dot" command: *append* to object

Example

```
sage: eq = 4*x^2 - 3*x + 1 == 0
sage: sols = solve(eq, x)
sage: len(sols)
2
```
(`len()` gives length of a collection)

```
sage: x1 = sols[0]
sage: x1
x == -1/8*I*sqrt(7) + 3/8
```
(oops! want only solution)

```
sage: x1 = sols[0].rhs()
sage: x1
-1/8*I*sqrt(7) + 3/8
```
(better)

Complex solutions?

1. `.real_part()`, `.imag_part()`
2. Can `round()` if desired

```
sage: sols = solve([x^5-3==0],x)
```

Complex solutions?

1. `.real_part()`, `.imag_part()`
2. Can `round()` if desired

```
sage: sols = solve([x^5-3==0],x)
sage: sols
[x == 3^(1/5)*e^(2/5*I*pi), x ==
3^(1/5)*e^(4/5*I*pi), x == 3^(1/5)*e^(-4/5*I*pi), x
== 3^(1/5)*e^(-2/5*I*pi), x == 3^(1/5)]
```

Complex solutions?

1. `.real_part()`, `.imag_part()`
2. Can `round()` if desired

```
sage: sols = solve([x^5-3==0],x)
sage: sols
[x == 3^(1/5)*e^(2/5*I*pi), x ==
3^(1/5)*e^(4/5*I*pi), x == 3^(1/5)*e^(-4/5*I*pi), x
== 3^(1/5)*e^(-2/5*I*pi), x == 3^(1/5)]
sage: sols[0].rhs().real_part()
1/4*sqrt(5)*3^(1/5) - 1/4*3^(1/5)
```

Complex solutions?

1. `.real_part()`, `.imag_part()`
2. Can `round()` if desired

```
sage:  sols = solve([x^5-3==0],x)
sage:  sols
[x == 3^(1/5)*e^(2/5*I*pi), x ==
3^(1/5)*e^(4/5*I*pi), x == 3^(1/5)*e^(-4/5*I*pi), x
== 3^(1/5)*e^(-2/5*I*pi), x == 3^(1/5)]
sage:  sols[0].rhs().real_part()
1/4*sqrt(5)*3^(1/5) - 1/4*3^(1/5)
sage:  sols[0].rhs().imag_part()
3^(1/5)*sin(2/5*pi)
```

Complex solutions?

1. `.real_part()`, `.imag_part()`
2. Can `round()` if desired

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sage: sols = solve([x^5-3==0],x)
sage: sols
[x == 3^(1/5)*e^(2/5*I*pi), x ==
3^(1/5)*e^(4/5*I*pi), x == 3^(1/5)*e^(-4/5*I*pi), x
== 3^(1/5)*e^(-2/5*I*pi), x == 3^(1/5)]
sage: sols[0].rhs().real_part()
1/4*sqrt(5)*3^(1/5) - 1/4*3^(1/5)
sage: sols[0].rhs().imag_part()
3^(1/5)*sin(2/5*pi)
sage: a, b = sols[0].rhs().real_part(),
sols[0].rhs().imag_part()
sage: round(a,5), round(b, 5)
(0.38495, 1.18476)
```

Solutions should solve

Extract second solution; substitute into equation

```
sage:  x2 = sols[1].rhs()
sage:  x2
1/8*I*sqrt(7) + 3/8
```

Solutions should solve

Extract second solution; substitute into equation

```
sage:  x2 = sols[1].rhs()
sage:  x2
1/8*I*sqrt(7) + 3/8
sage:  eq(x=x2)
4*(1/8*I*sqrt(7) + 3/8)^2
 - 3/8*I*sqrt(7) - 1/8 == 0
```

(need to expand product)

Solutions should solve

Extract second solution; substitute into equation

```
sage: x2 = sols[1].rhs()

sage: x2
1/8*I*sqrt(7) + 3/8

sage: eq(x=x2)
4*(1/8*I*sqrt(7) + 3/8)^2
 - 3/8*I*sqrt(7) - 1/8 == 0
```

(need to expand product)

```
sage: expand(eq(x=x2))
0 == 0
```

Calculus: 1 picture = 1000 words

Let's diagram the critical points to $f(x) = x^3 - 4x + 1$.

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```
sage: f(x) = x^3 - 4*x + 1
```

Calculus: 1 picture = 1000 words

Let's diagram the critical points to $f(x) = x^3 - 4x + 1$.

```
sage: f(x) = x^3 - 4*x + 1
sage: df(x) = diff(f)
sage: crit_pts = solve(df(x), x)
sage: crit_pts
[x == -2/3*sqrt(3), x == 2/3*sqrt(3)]
```

Calculus: 1 picture = 1000 words

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```
sage: f(x) = x^3 - 4*x + 1
sage: df(x) = diff(f)
sage: crit_pts = solve(df(x), x)
sage: crit_pts
[x == -2/3*sqrt(3), x == 2/3*sqrt(3)]
sage: crit_pts = [ a.rhs() for a in crit_pts ]
sage: crit_pts
[-2/3*sqrt(3), 2/3*sqrt(3)]
```

Calculus: 1 picture = 1000 words

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sage: crit_pts
[x == -2/3*sqrt(3), x == 2/3*sqrt(3)]
sage: crit_pts = [ a.rhs() for a in crit_pts ]
sage: crit_pts
[-2/3*sqrt(3), 2/3*sqrt(3)]
sage: p = plot(f, min(crit_pts) - 1,
                  max(crit_pts) + 1, color='black',
                  thickness=2)
sage: p += sum(point((a, f(a)), color='red',
                  pointsize=90) for a in crit_pts)
```

Calculus: 1 picture = 1000 words

Let's diagram the critical points to $f(x) = x^3 - 4x + 1$.

```
sage: f(x) = x^3 - 4*x + 1
sage: df(x) = diff(f)
sage: crit_pts = solve(df(x), x)
sage: crit_pts
[x == -2/3*sqrt(3), x == 2/3*sqrt(3)]
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[-2/3*sqrt(3), 2/3*sqrt(3)]
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                   pointsize=90) for a in crit_pts)
sage: p
```
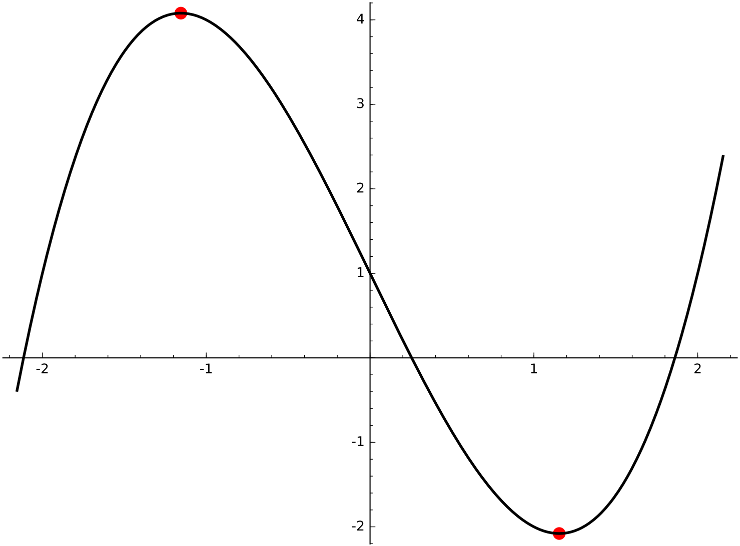

Calculus: 1 picture = 1000 words



Solving linear inequalities

Just like solve equations, except solution is list of lists

```
sage:  solve((x-3)*(x-1)*(x+1)*(x+3) >= 0, x)
[[x <= -3], [x >= -1, x <= 1], [x >= 3]]
```

Solving linear inequalities

Just like solve equations, except solution is list of lists

```
sage:  solve((x-3)*(x-1)*(x+1)*(x+3) >= 0, x)
[[x <= -3], [x >= -1, x <= 1], [x >= 3]]
```

Each sublist represents interval of solutions:

- `[x <= -3]` $\iff (-\infty, -3]$
- `[x >= -1, x <= 1]` $\iff [-1, \infty) \cap (-\infty, 1] \iff [-1, 1]$
- `[x >= 3]` $\iff [3, \infty)$

Systems of linear equations

- system of linear, multivariate equations
- can always be solved *exactly*
- zero, one, or infinitely many solutions
- solution is a list of solutions

No solution

```
sage:  var('y')
(y)
sage:  solve([x + y == 1,
              x + y == 0],
             [x,y])
```

...output cut...

```
[]
```

One solution

```
sage:  var('z')
(z)
sage:  solve([3*x - 4*y + z == 1,
              2*x - 3*y + 4*z == 2,
              7*x + 10*y - 39*z == 1],
             [x,y,z])
[[x == (3/2), y == 1, z == (1/2)]]
```

Infinitely many solutions

```
sage:  solve([3*x - 4*y + z == 1,
              2*x - 3*y + 4*z == 2,
              -6*x + 8*y - 2*z == -2],
             [x,y,z])
[[x == 13*r1 - 5, y == 10*r1 - 4, z == r1]]
```

`r1`?!? What is `r1`?

`r1` is a *parameter* that can take infinitely many values

```
[[x == 13*r1 - 5, y == 10*r1 - 4, z == r1]]
```

corresponds to

$$x = 13t - 5, \quad y = 10t - 4, \quad z = t.$$

Example

$t = 0$?

- $x = -5$, $y = -4$, $z = 0$
- Substitute into system:

$$\begin{aligned} 3(-5) - 4(-4) + 0 &= 1 \\ 2(-5) - 3(-4) + 4(0) &= 2 \\ -6(-5) + 8(-4) - 2(0) &= -2. \end{aligned}$$

Extract and test

```
sage: eq1 = 3*x - 4*y + z == 1
sage: eq2 = 2*x - 3*y + 4*z == 2
sage: eq3 = -6*x + 8*y - 2*z == -2
sage: sols = solve([eq1, eq2, eq3], [x,y,z])
```

sols is a list of lists. . .

```
sage: sol1 = sols[0]
sage: x1 = sol1[0].rhs()
sage: y1 = sol1[1].rhs()
sage: z1 = sol1[2].rhs()
sage: x1,y1,z1
(13*r2 - 5, 10*r2 - 4, r2)
sage: eq1(x=x1,y=y1,z=z1)
1 == 1
```

Outline

Why approximate?

- Exact solutions often… *complicated*

$$-\frac{1}{2}\cdot\sqrt[3]{\frac{\sqrt{3713}}{54}+\frac{7}{6}}\cdot\left(1+i\sqrt{3}\right)+\frac{-2+2i\sqrt{3}}{9}\cdot\sqrt[3]{\frac{\sqrt{3713}}{54}+\frac{7}{6}}$$

- Approximate solutions easier to look at, manipulate
  $-0.8280018073-0.8505454986i$
- Approximation often *much*, ***much*** faster!
  - except when approximation fails
    - bad condition numbers
    - rounding errors
    - inappropriate algorithm (real solver, complex roots)

The `find_root()` command

`find_root(`*equation*`,` *xmin*`,` *xmax*`)` where

- *equation* has a root between real numbers *xmin* and *xmax*
- reports an error if no root exists
- this is a *real solver:* looks for *real* roots
- uses Scipy package

Example

```
sage: find_root(x^5+2*x+1==0,-10,0)
-0.486389035934542970
sage: find_root(x^5+2*x+1==0,0,10)
```

...output cut...

```
RuntimeError: f appears to have no zero on the interval
```

The `.roots()` command

polynomial`.roots()` ordinarily finds exact roots of a polynomial, along with multiplicities

- reports error if cannot find explicit roots
- approximate real roots: option `ring=RR`
- approximate complex roots: option `ring=CC`
- uses Scipy package
- “multiplicity” = “shape” of root
    - linear, quadratic, cubic, ...

Ring?!?

field addition, multiplication as in rational, real, complex numbers

Ring?!?

field addition, multiplication as in rational, real, complex numbers

ring addition, multiplication common to integers, matrices, and fields

- $+$ as usual
- $\times$ weird sometimes
  - $ab \neq ba$ — matrices
  - no $1/a$ even if $a \neq 0$ — integers, matrices
  - $ab = 0$ but $a, b \neq 0$ — matrices

Exact example

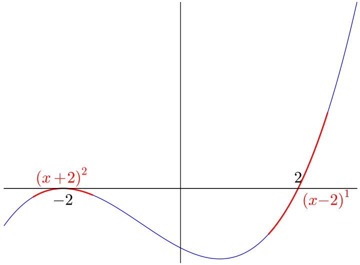
```
sage: p = x^3 + 2*x^2 - 4*x - 8
sage: p.roots()
[(2, 1), (-2, 2)]
```

roots are 2 (mult. 1) and -2 (mult. 2)

Exact example

```
sage: p = x^3 + 2*x^2 - 4*x - 8
sage: p.roots()
[(2, 1), (-2, 2)]
```

roots are 2 (mult. 1) and -2 (mult. 2)



see if you can make Sage produce this image!

Approximate example

```
sage: p = x^5 + 2*x + 1
sage: p.roots()
```

...output cut...

```
RuntimeError: no explicit roots found
```

Approximate example

```
sage: p = x^5 + 2*x + 1
sage: p.roots()
...output cut...
RuntimeError: no explicit roots found
sage: p.roots(ring=RR)
[(-0.486389035934543, 1)]
```

root approximately $-.486389$ *w/multiplicity 1*

Approximate example

```
sage: p = x^5 + 2*x + 1
sage: p.roots()
...output cut...
RuntimeError: no explicit roots found
sage: p.roots(ring=RR)
[(-0.486389035934543, 1)]
```

root approximately $-.486389$ *w/multiplicity 1*

Fundamental Theorem of Algebra

Every polynomial of degree n has n complex roots.

Where are the other 4 roots?

Extract and use complex roots

```
sage:  sols = p.roots(ring=CC)
```

How can we extract roots?

Extract and use complex roots

```
sage:  sols = p.roots(ring=CC)
```

How can we extract roots?

`sols` is a list of tuples $(root, \quad multiplicity)$:
need to extract tuple, *then* root

```
sage:  x0 = sols[0]
sage:  x0
(-0.486389035934543, 1)
sage:  x0 = sols[0][0]
sage:  x0
-0.486389035934543
sage:  x1 = sols[1][0]
sage:  x1
-0.701873568855862 - 0.879697197929823*I
```

want first root

oops! this is the tuple!

root is first element of tuple

want second root

What is going on here?

`sols`

0	0	$-0.486389\ldots$	(approximation)
	1	1	(multiplicity)
1	0	$-0.701873\ldots - 0.879697\ldots i$	(approximation)
	1	1	(multiplicity)
$\vdots$		$\vdots$	

What is going on here?

`sols`

0	0	$-0.486389\ldots$	(approximation)
	1	1	(multiplicity)
1	0	$-0.701873\ldots - 0.879697\ldots i$	(approximation)
	1	1	(multiplicity)
⋮		⋮	

- first bracket: gets solution
- each solution is a tuple
  - second bracket: gets information about solution

    `[0]` approximation

    `[1]` multiplicity

Outline

Summary

- distinguish = (assignment) and == (equality)
- Sage can find *exact* or *approximate* roots
- `solve()` finds exact solutions
  - not all equations can be solved exactly
  - systems of linear equations always exact
  - extract using `[ ]` and `.rhs()`
- `find_root()` approximates real roots on an interval
  - error if no roots on interval
- `.roots(ring=...)` approximates roots
  - `RR` for real roots only; `CC` for all complex roots
  - append to polynomial or equation