MAT 305: Mathematical Computing

Linear algebra

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Outline

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What is a vector?

Physics A **vector** has *direction* and *size*.

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Mathematics A **vector** is a list of numbers.

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Mathematics A **vector** is a list of numbers.

Example

"30° at 400ft/sec" vs. $\begin{pmatrix} 346 & 200 \end{pmatrix}$

Vectors in Sage

`vector(`*ring*`,` *entries*`)` where

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- default ring: appropriate to entries ($\mathbb{Z}$ for integers)

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Example

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sage:  u = vector([0, 2, 2, 0])
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```

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Example

```
sage: u = vector([0, 2, 2, 0])
sage: v = vector([1/3, 0, -1, 2])
sage: u.parent().base_ring()
Integer Ring
sage: v.parent().base_ring()
Rational Field
```

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Integer Ring
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Rational Field
sage: u + v
(1/3, 2, 1, 2)
```

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sage:  u.parent().base_ring()
Integer Ring
sage:  v.parent().base_ring()
Rational Field
sage:  u + v
(1/3, 2, 1, 2)
sage:  u*v, u.norm()
(-2, 2*sqrt(2))
```

Dot product!

You can plot vectors!

`v.plot()`, with optional arguments:

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Example

Illustration of vector arithmetic:

```
sage: u = vector([1,2])
sage: v = vector([3,-1])
sage: u.plot(color='red')
        + v.plot(color='blue',start=u)
        + (u+v).plot(color='purple')
```

You can plot vectors!

Example

Illustration of vector arithmetic:

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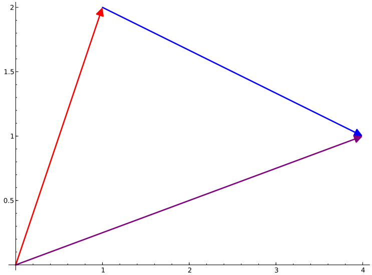

Outline

The `matrix()` command

`matrix(` *ring* `,` *#rows* `,` *#cols* `,` entries`)` where

- *ring* (optional) an appropriate algebraic ring
- *#rows*, *#cols* (optional) number of rows and columns (default depends on *entries*; no *entries* $\Longrightarrow$ 0×0 matrix)
- *entries* (optional) is one of
  - a list of entries, from northwest corner to southeast (if *#rows*, *#cols* specified)
  - a list of row vectors
  - none specified? all entries 0

Example matrices

```
sage:  MZ = matrix(ZZ,3,3)
sage:  MZ
[0 0 0]
[0 0 0]
[0 0 0]
```

Example matrices

```
sage: MZ = matrix(ZZ,3,3)
sage: MZ
[0 0 0]
[0 0 0]
[0 0 0]
sage: MR = matrix(RR,[[1,2,3],[3,2,1],[1,1,2]])
sage: MR
[1.00000000000000 2.00000000000000 3.00000000000000]
[3.00000000000000 2.00000000000000 1.00000000000000]
[1.00000000000000 1.00000000000000 2.00000000000000]
```

Example matrices

```
sage:  MZ = matrix(ZZ,3,3)
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[0 0 0]
[0 0 0]
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sage:  MR
[1.00000000000000 2.00000000000000 3.00000000000000]
[3.00000000000000 2.00000000000000 1.00000000000000]
[1.00000000000000 1.00000000000000 2.00000000000000]
sage:  MS = matrix(SR,[[x**2 + 1, 0, 0],
       [x + I, 1, 0]])
sage:  MS
[x^2 + 1       0       0]
[  x + I       1       0]
```

Help yourself read

Good idea to put rows in different lines

```
sage:  MR = matrix(RR, [
        [1,2,3],
        [3,2,1],
        [1,1,2]
      ])
sage:  MR
[1.00000000000000 2.00000000000000 3.00000000000000]
[3.00000000000000 2.00000000000000 1.00000000000000]
[1.00000000000000 1.00000000000000 2.00000000000000]
```

Accessing matrix entries

Matrix a list of lists $\Longrightarrow$ $M_{i,j} = \texttt{M[i,j]}$

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Example

(counting starts from 0)

```
sage: MS[1,0]
x+I
sage: MS[0,2] = x - I
sage: MS
[x^2 + 1       0       x - I]
[  x + I       1           0]
```

Submatrices

- `M.submatrix(`i, j, m, n`)` gives
  - $m \times n$ submatrix of M
  - northwest corner is in row i, column j
- `M.augment(`A`)` gives $(M|A)$

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Example

```
sage:  MZ[1,1] = 1
sage:  MZ.submatrix(1,1,2,2)
[ 1 0]
[ 0 0]
```

Basic matrix operations

“dot” command	mathematics
`M.det()`	determinant
`M.inverse()`	
`M.transpose()`	
`M.eigenvalues()`	
`M.eigenvectors_right()`	right eigenvectors*
`M.eigenvectors_left()`	left eigenvectors*
`M.echelon_form()`	echelon formof unchanged M
`M.echelonize()`	change M to echelon form
`M.ncols()`	number of columns
`M.nrows()`	number of rows

*“right eigenvectors” are usual “eigenvectors”

Row arithmetic

“dot” command	mathematics
`M.set_row_to_multiple_of_row(i,j,a)`	set row i to a times row j*
`M.add_multiple_of_row(i,j,a)`	add a times row j to row i*
`M.swap_rows(i,j)`	swap rows i, j
`M.swap_columns(i,j)`	swap columns i, j

*row i changes; row j remains the same

Example: find inverse of matrix

Sage has a `.inverse()` command, but suppose you want to see steps…?

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Algorithm from High School Algebra II!

algorithm Compute inverse
inputs
M, an invertible matrix over a field
outputs
M^{-1}
do
Let $n = \dim(M)$
Let A be augmented matrix $(M \mid I_n)$
Triangularize A
return rightmost $n \times n$ submatrix of A

Try it!

```
sage:  MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
               [8, 3, 1, 2], [0, 1, 2, 3]])
```

Augment `MZ` *by* I_4

Try it!

```
sage:  MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
                 [8, 3, 1, 2], [0, 1, 2, 3]])
```

Augment `MZ` *by* I_4

To create I_4*, can set diagonal entries of zero matrix to 1…*

```
sage:  I4 = matrix(4,4)
sage:  for i in range(4):
           I4[i,i] = 1
sage:  I4
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

Try it!

```
sage: MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
                   [8, 3, 1, 2], [0, 1, 2, 3]])
```

Augment `MZ` *by* I_4

. . . *or use* `identity_matrix()` *command*

```
sage: I4 = identity_matrix(4)
[1 0 0 0]
[0 1 0 0]
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```

Try it!

```
sage:  MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
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```

Augment `MZ` *by* I_4

…*or use* `identity_matrix()` *command*

```
sage:  I4 = identity_matrix(4)
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
sage:  A = MZ.augment(I4)
sage:  A
[1 2 3 4 1 0 0 0]
[0 2 2 3 0 1 0 0]
[8 3 1 2 0 0 1 0]
[0 1 2 3 0 0 0 1]
```

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sage:  MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
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```

First column: eliminate non-zero in row 3

Try it!

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```

First column: eliminate non-zero in row 3

```
sage:  A.add_multiple_of_row(2,0,-8)
sage:  A
[   1   2   3   4   1   0   0   0]
[   0   2   2   3   0   1   0   0]
[   0 -13 -23 -30  -8   0   1   0]
[   0   1   2   3   0   0   0   1]
```

Try it!

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```

Second column: swap row w/pivot to row 2,
eliminate other non-zeros

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```
sage:  A.swap_rows(1,3)
sage:  A.add_multiple_of_row(0,1,-2)
sage:  A.add_multiple_of_row(2,1,13)
sage:  A.add_multiple_of_row(3,1,-2)
sage:  A
[  1   0  -1  -2   1   0   0  -2]
[  0   1   2   3   0   0   0   1]
[  0   0   3   9  -8   0   1  13]
[  0   0  -2  -3   0   1   0  -2]
```

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sage: A
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[  0   0  -2  -3   0   1   0  -2]
```

Third column: need pivot multiply row 3 by 1/3

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```
sage: A.set_row_to_multiple_of_row(2,2,1/3)
...
TypeError: Multiplying row by Rational Field
element cannot be done over Integer Ring, use
change_ring or with_row_set_to_multiple_of_row
instead.
```

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```

Uh-oh! No multiplicative inverses in default ring! ($\mathbb{Z}$)
Change to $\mathbb{Q}$ *and proceed.*

Try it!

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Uh-oh! No multiplicative inverses in default ring! ($\mathbb{Z}$)
Change to $\mathbb{Q}$ *and proceed.*

```
sage:  A = A.change_ring(QQ)
sage:  A
[  1   0  -1  -2   1   0   0  -2]
[  0   1   2   3   0   0   0   1]
[  0   0   3   9  -8   0   1  13]
[  0   0  -2  -3   0   1   0  -2]
```

Looks the same, but it's not.

Try it!

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sage:  A.set_row_to_multiple_of_row(2,2,1/3)
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[    0     0    -2    -3     0     1     0    -2]
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Third column: eliminate other non-zeros

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```
sage:  A.add_multiple_of_row(0,2,1)
sage:  A.add_multiple_of_row(1,2,-2)
sage:  A.add_multiple_of_row(3,2,2)
sage:  A
[     1      0      0      1   -5/3      0    1/3    7/3]
[     0      1      0     -3   16/3      0   -2/3  -23/3]
[     0      0      1      3   -8/3      0    1/3   13/3]
[     0      0      0      3  -16/3      1    2/3   20/3]
```

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```

Fourth column: need pivot
multiply row 4 by 1/3

Try it!

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```
sage:  A.set_row_to_multiple_of_row(3,3,1/3)
sage:  A
[     1     0     0     1  -5/3     0   1/3   7/3]
[     0     1     0    -3  16/3     0  -2/3 -23/3]
[     0     0     1     3  -8/3     0   1/3  13/3]
[     0     0     0     1 -16/9   1/3   2/9  20/9]
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Fourth column: eliminate other non-zeros

```
sage: A.add_multiple_of_row(0,3,-1)
sage: A.add_multiple_of_row(1,3,3)
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sage: A
[     1     0     0     0   1/9  -1/3   1/9   1/9]
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[     0     0     1     0  -8/3    -1  -1/3  -7/3]
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[   0    0    1    0  -8/3    -1  -1/3  -7/3]
[   0    0    0    1 -16/9   1/3   2/9  20/9]
```

Have inverse! extract, test

Try it!

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Try it!

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```

Have inverse! extract, test

```
sage: Minv = A.submatrix(0,4,4,4)
sage: Minv * M
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

Other tools

Need another computation w/M? Remember:

- `M.<tab>` states all tools for M
- `M.<command>?` states help for `command`
- `M.<command>??` lists source code for `command`

Outline

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Manipulating graphics

Can manipulate point (x, y) using matrix arithmetic:

- let $\mathbf{v} = (x, y)$ be vector
- let M be matrix of special form
- $M\mathbf{v}$ gives new point

Useful matrix forms

type	scaling	rotation	reflection
form	$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$	$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$	$\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$
effect	rescales (x,y) to (sx,sy)	rotates (x,y) through angle α	reflects (x,y) across line w/slope $^{1-\cos\beta}/_{\sin\beta}$

Example; rotate a polygon

Start with star whose points are at angles $4\pi i/5$ for $i = 0, 1, \ldots, 4$.

```
sage: V = [
        vector( ( cos(4*i*pi/5), sin(4*i*pi/5) ) )
        for i in range(5)
          ]
sage: polygon(U)
```

Example; rotate a polygon

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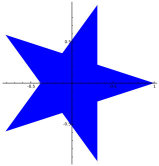

A little off… $\pi/10$ maybe?

Use a rotation matrix

try $\alpha = \pi/10$

```
sage:  M = matrix((
                (cos(pi/10), sin(pi/10)),
                (-sin(pi/10), cos(pi/10))
              ))
```

Use a rotation matrix

try $\alpha = \pi/10$

```
sage:  M = matrix((
                  (cos(pi/10), sin(pi/10)),
                  (-sin(pi/10), cos(pi/10))
                ))
sage:  U = [M*v for v in V]
sage:  polygon(U)
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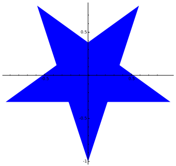

Oops! upside-down!

Rotate the *other* way

try $\alpha = -\pi/10$

```
sage:  M = matrix((
                (cos(-pi/10), sin(-pi/10)),
                (-sin(-pi/10), cos(-pi/10))
              ))
```

Rotate the *other* way

try $\alpha = -\pi/10$

```
sage:  M = matrix((
              (cos(-pi/10), sin(-pi/10)),
              (-sin(-pi/10), cos(-pi/10))
            ))
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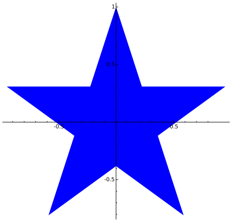

How about an animation?

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```
sage:  U = copy(V)
sage:  M = matrix((
     (round(cos(2*pi/60),5), round(sin(2*pi/60),5)),
     (round(-sin(2*pi/60),5), round(cos(2*pi/60),5))
))
sage:  frames = []
sage:  for each in range(60):
         U = [ M*u for u in U ]
         frames.append(
            polygon(U, xmin=-1, max=1, ymin=-1,
ymax=1)
         )
sage:  show(animate(frames), delay=6)
```

Outline

Eigenvectors and eigenvalues

An **eigenvector** $\mathbf{x}$ of a matrix M with **eigenvalue** λ satisfies

$$M\mathbf{x} = \lambda\mathbf{x}$$

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Example

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

verification in Sage:

```
sage:  M = matrix(2,2,[0,2,2,0])
sage:  v = vector([1,-1])
sage:  M*v
(-2, 2)
```

Easy to find in Sage

```
sage:  M = matrix(2,2,[0,2,2,0])
sage:  M.eigenvectors_left()
[(2, [ (1, 1) ], 1), (-2, [ (1, -1) ], 1)]
```

What does this tell us?

- $\mathbf{e}_1 = (1,1)$ is eigenvec w/eigenval 2, mult 1
- $\mathbf{e}_2 = (1,-1)$ is eigenvec, w/eigenval -2, mult 1

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In other words,

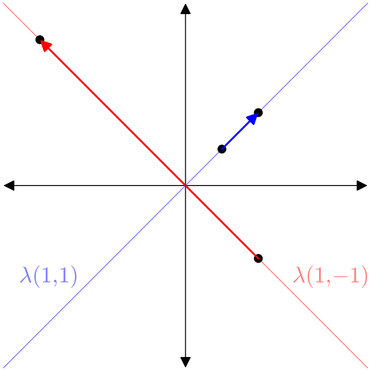
- $M\mathbf{e}_1 = 2\mathbf{e}_1$
- $M\mathbf{e}_2 = -2\mathbf{e}_2$

Verify in Sage

Geometric interpretation

$M\mathbf{x} = \lambda\mathbf{x}$

- $\lambda\mathbf{x}$ on same *line through origin* as $\mathbf{x}$
  - $\lambda > 0$? same direction
  - $\lambda < 0$? opposite direction
- $\lambda\mathbf{x}$ different *size* from $\mathbf{x}$



Outline

Summary

- Sage does matrices
  - over fields *and* rings
  - symbolic ring! explore!
  - can change base ring

- You can solve some sophisticated problems using matrices on Sage