MAT 305: Mathematical Computing

Decision-making

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Spring 2019

Outline

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Decision making?

A function may have to act in different ways, depending on the arguments.

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A function may have to act in different ways, depending on the arguments.

Example

Piecewise functions:

$$f(x) = \begin{cases} f_1(x), & x \in (a_0, a_1) \\ f_2(x), & x \in [a_1, a_2). \end{cases}$$

If $x \in (a_0, a_1)$, then $f(x) = f_1(x)$;
if $x \in [a_1, a_2)$, then $f(x) = f_2(x)$.

Decision making?

A function may have to act in different ways, depending on the arguments.

Example

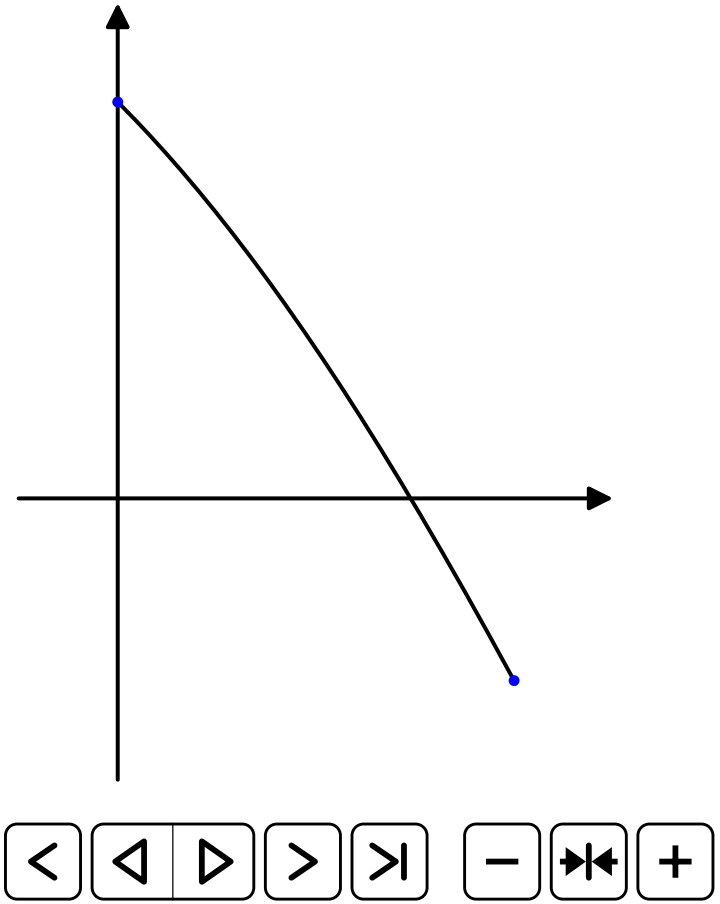
Deciding concavity:

$$\text{If } f''(a) > 0, \text{ then } f \text{ is concave up at } x = a;$$
$$\text{if } f''(a) < 0, \text{ then } f \text{ is concave down at } x = a.$$

Method of bisection

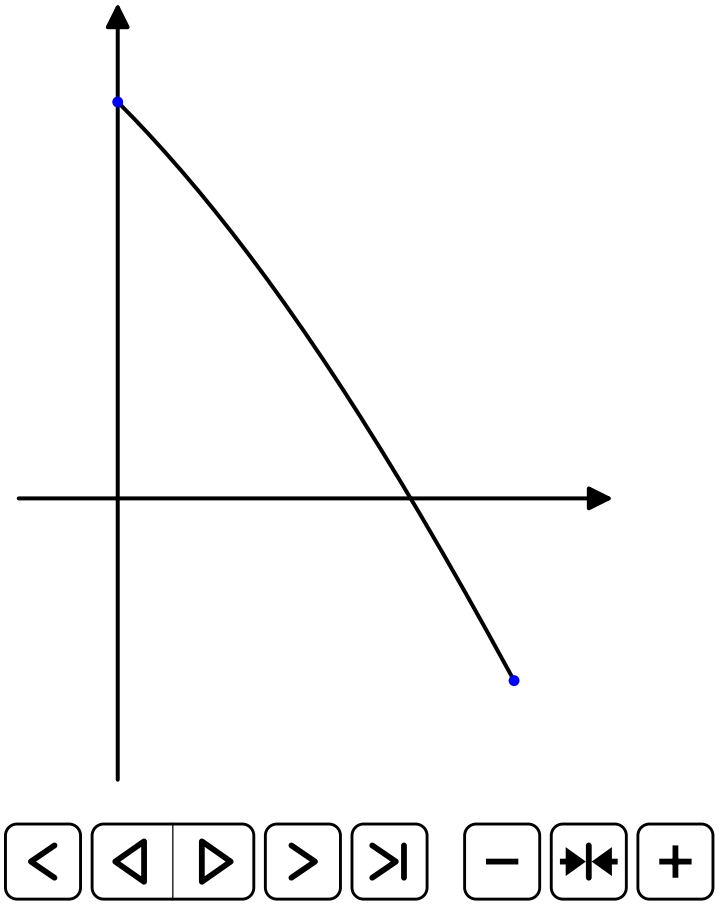
From Calculus I:

- We want to find the root of a *continuous* polynomial.
- We know it lies on somewhere on an interval.
- By repeatedly halving the interval, we can narrow down to something "accurate"



Pseudocode style

repeat n times
  let $c = {}^{a+b}/_{2}$
  if $f(a), f(c)$ same sign
    replace a by c
  otherwise
    replace b by c



`if` statements

`if` *condition* `:`
  if-statement1
  if-statement2
  ...
non-if statement1

where

- *condition*: expression that evaluates to `True` or `False`
- *condition* `True?` *if-statement1*, *if-statement2*, ... performed
  - proceed eventually to *non-if statement1*
- *condition* `False?` *if-statement1*, *if-statement2*, ... skipped
  - proceed immediately to *non-if statement1*

Example

```
sage:  f(x) = cos(x)

sage:  ddf(x) = diff(f,2)

sage:  if ddf(3*pi/4) > 0:
           print 'concave up at', 3*pi/4
concave up at 3/4*pi
```

`if-else` statements

`if` *condition*`:`
  if-statement1
  ...
`else:`
  else-statement1
  ...
non-if statement1

where

- *condition* `True?` *if-statement1*, ... performed
  - *else-statement1*, ... skipped
- condition `False?` *else-statement1*, ... performed
  - *statement1*, ... skipped
- proceed sooner or later to *non-if statement1*

`if-elif-else` statements

```
if condition1:
    if-statement1
    ...
elif condition2:
    elif1-statement1
    ...
elif condition3:
    elif2-statement1
    ...
...
else:
    else-statement1
    ...
non-if statement1
```

Pseudocode for `if-elif-else`

if *condition1*
    if-statement1
    …
else if *condition2*
    elseif1-statement1
    …
else if *condition3*
    elseif2-statement1
    …
…
else
    else-statement1
    …

Notice:

- indentation
- no colons
- **else if**, not **elif**

Example: concavity

Write a Sage function that tests whether a function f is concave up or down at a given point. Have it return the string `'concave up'`, `'concave down'`, or `'neither'`.

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Different choices $\Longrightarrow$ need to decide! $\Longrightarrow$ **if**

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Write a Sage function that tests whether a function f is concave up or down at a given point. Have it return the string `'concave up'`, `'concave down'`, or `'neither'`.

Different choices $\Longrightarrow$ need to decide! $\Longrightarrow$ **if**

Start with pseudocode.

- inputs needed?
- output expected?
- what to do?
  - step by step
  - *Divide et impera!* **Divide and conquer!**

Pseudocode for Example

algorithm *check_concavity*
inputs

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$a \in \mathbb{R}$

$f(x)$, a twice-differentiable function at $x = a$

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`'concave up'` if f is concave up at $x = a$

`'concave down'` if f is concave down at $x = a$

`'neither'` otherwise

do

Pseudocode for Example

algorithm *check_concavity*

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$a \in \mathbb{R}$

$f(x)$, a twice-differentiable function at $x = a$

outputs

`'concave up'` if f is concave up at $x = a$

`'concave down'` if f is concave down at $x = a$

`'neither'` otherwise

do

if $f''(a) > 0$

return `'concave up'`

else if $f''(a) < 0$

return `'concave down'`

else

return `'neither'`

Try it!

```
sage:  def check_concavity(a, f, x):
         ddf = diff(f, x, 2)
         if ddf(x=a) > 0:
           return 'concave up'
         elif ddf(x=a) < 0:
           return 'concave down'
         else:
           return 'neither'
```

Try it!

```
sage:  def check_concavity(a, f, x):
         ddf = diff(f, x, 2)
         if ddf(x=a) > 0:
           return 'concave up'
         elif ddf(x=a) < 0:
           return 'concave down'
         else:
           return 'neither'
sage:  check_concavity(3*pi/4, cos(x), x)
'concave up'
sage:  check_concavity(pi/4, cos(x), x)
'concave down'
```

More interesting example

Use the Method of Bisection to approximate a root of $\cos x - x$ on the interval $[0, 1]$, correct to the hundredths place.

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Use the Method of Bisection to approximate a root of $\cos x - x$ on the interval $[0, 1]$, correct to the hundredths place.

???

Method of Bisection?

The Method of Bisection is based on:

Theorem (Intermediate Value Theorem)

If

- f *is a continuous function on* $[a, b]$, *and*
- $f(a) \neq f(b)$,

then

- *for any* y *between* $f(a)$ *and* $f(b)$,
- $\exists c \in (a, b)$ *such that* $f(c) = y$.

Continuous?

f *continuous* at $x = a$ if

- can evaluate limit at $x = a$ by computing $f(a)$, or
- can draw graph without lifting pencil

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Upshot: To find a root of a continuous function f, start with two x values a and b such that $f(a)$ and $f(b)$ have different signs, then bisect the interval.

Back to the example…

Check hypotheses…

- $f(x) = \cos x - x$
  - x, $\cos x$ continuous
  - difference of continuous functions also continuous
  - $\therefore$ f continuous
- $a = 0$ and $b = 1$
  - $f(a) = 1 > 0$
  - $f(b) \approx -0.4597 < 0$

Intermediate Value Theorem applies: can start Method of Bisection.

How to solve it?

Idea: Interval endpoints a and b are not close enough as long as their digits differ through the hundredths place.

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"Halve" the interval? Pick the half containing a root!

Method of bisection

algorithm *Method_of_Bisection*
inputs
$a, b \in \mathbb{R}$
f, a continuous function on $[a,b]$
n, number of bisections
outputs
$[c,d] \subseteq [a,b]$ such that
$d-c = (b-a)/2^n$, and
a root of f lies in $[c,d]$
do
let $c=a$, $d=b$
repeat n times
let $e=(c+d)/2$
if $f(c)$ and $f(e)$ have same sign
replace c by e
else
replace d by e
return $[c,d]$

Sage code

```
def method_of_bisection(a, b, n, f, x=x):
  c, d = a, b
  f(x) = f
  for each in range(n):
    e = (c + d) / 2
    if f(c)*f(e) > 0:
      c = e
    else:
      d = e
  return (c,d)
```

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    if f(c)*f(e) > 0:
      c = e
    else:
      d = e
  return (c,d)
```

```
sage:  method_of_bisection(0, 1, 20, cos(x) - x)
(0, 1/1048576)
```

Outline

Boolean algebra

Boolean algebra operates on only two values: {`True`, `False`}.

…or {1, 0} if you prefer

…or {Yes, No} if you prefer

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Basic operations:

- **not** x
  - `True` iff x is `False`
- x **and** y
  - `True` iff both x and y are `True`
- x **or** y *("inclusive" or)*
  - `True` iff
    - x is `True`; or
    - y is `True`; or
    - both x and y are `True`

Example: and, or

```
sage:  5 > 4
True
```

obvious enough

```
sage:  5 < 4
False
```

```
sage:  (5 > 4) or (5 < 4)
True
```

because at least one is `True` ($5 > 4$)

```
sage:  (5 > 4) and (5 < 4)
False
```

because one is `False`

Example: `not`

```
sage:  4 > 4
False
```

obvious enough

```
sage:  not (4 > 4)
True
```

```
sage:  not ((5 > 4) or (4 < 5))
False
```

we have `(not True)`

```
sage:  not (4 == 5)
True
```

we have `(not False)`

Equality and inequalities

Recall: = and == are not the same

- `x = y` assigns value of `y` to `x`
- `x == y` compares values of `x`, `y`, reports `True` or `False`

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For inequalities,

- `x != y` compares `x`, `y`
  - `True` iff `not (x == y)`
- `x > y`, `x < y` have usual meanings

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Recall: `=` and `==` are not the same

- `x = y` assigns value of `y` to `x`
- `x == y` compares values of `x`, `y`, reports `True` or `False`

For inequalities,

- `x != y` compares `x`, `y`
  - `True` iff `not (x == y)`
- `x > y`, `x < y` have usual meanings
- $x \geq y$? use `x >= y`
  - `True` iff `not (x < y)`
- $x \leq y$? use `x <= y`
  - `True` iff `not (x > y)`

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Break what's fixed!

If a loop arrives at an answer we like, earlier than we expect, `break` tells Sage to break out of the (innermost) loop and continue

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Example (Method of bisection)

If we find an *actual root*, no point in continuing the loop.

Sage code

```
def method_of_bisection(a, b, n, f, x=x):
  c, d = a, b
  f(x) = f
  for each in range(n):
    e = (c + d) / 2
    if f(c)*f(e) > 0:
      c = e
    elif f(e) == 0:
      c = d = e
      break
    else:
      d = e
  return (c,d)
```

Example

```
sage:  method_of_bisection(-1, 1, 20, 4*x + 3)
(-3/4, -3/4)
```

Example

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sage:  method_of_bisection(-1, 1, 20, 4*x + 3)
(-3/4, -3/4)
```

Actual result found in 3 steps, not 20!

$$(-1, 1), (-1, 0), (-1, -1/2), (-1, -3/4), (-7/8, -3/4), (-13/16, -3/4), \ldots$$

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Exceptions

Sage classifies errors according to certain types

- SyntaxError
- TypeError
- ZeroDivisionError

...

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...

The management of special errors is called **exception handling.**

try/except

Think an exception can occur? use a `try/except` block.

```
sage:  try:
           a = 1/0
       except ZeroDivisionError:
           a = Infinity
sage:  a
+Infinity
```

try/except

Think an exception can occur? use a `try/except` block.

```
sage:  try:
           a = 1/0
       except ZeroDivisionError:
           a = Infinity
sage:  a
+Infinity
```

Need not specify exception type.

```
sage:  try:
           a = 1/0
       except:
           a = Infinity
sage:  a
+Infinity
```

General form

```
try:
  try_statement_1
  try_statement_2
  ...
```

General form

```
try:
  try_statement_1
  try_statement_2
  ...
except exception_1:
  exception1_statement_1
  exception1_statement_2
  ...
```

General form

```
try:
  try_statement_1
  try_statement_2
  ...
except exception_1:
  exception1_statement_1
  exception1_statement_2
  ...
except exception_2:
  exception2_statement_1
  exception2_statement_2
  ...
```

General form

```
try:
  try_statement_1
  try_statement_2
  ...
except exception_1:
  exception1_statement_1
  exception1_statement_2
  ...
except exception_2:
  exception2_statement_1
  exception2_statement_2
  ...
...
```

Example

A while back you had to implement a *normal line*. One way to do it:

```
sage:  def normal_line(a, f, x=x):
           f(x) = f
           df(x) = diff(f)
           m=1/df(a)
           return m*(x - a) + f(a)
```

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This has an "obvious" error. What is it?

Example

A while back you had to implement a *normal line*. One way to do it:

```
sage:  def normal_line(a, f, x=x):
           f(x) = f
           df(x) = diff(f)
           m=1/df(a) # Possible division by 0
           return m*(x - a) + f(a)
```

This has an "obvious" error. What is it?

Catch, handle the error

Problem arises when $\Delta x = 0$:

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- implies vertical line

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- has form $x = a$

Catch, handle the error

Problem arises when $\Delta x = 0$:

- implies vertical line
- has form $x = a$
- return this!

In code

```
sage:  def normal_line(a, f, x=x):
           f(x) = f
           df(x) = diff(f)
           try:
             m = 1/df(a)
             result = y == m*(x - a) + f(a)
           except ZeroDivisionError:
             result = x == a
           return result
```

In code

```
sage:  def normal_line(a, f, x=x):
         f(x) = f
         df(x) = diff(f)
         try:
           m = 1/df(a)
           result = y == m*(x - a) + f(a)
         except ZeroDivisionError:
           result = x == a
         return result
sage:  normal_line(0, x^2)
x == 0
```

This can raise an error, too

```
sage:  normal_line(1, x^2)
NameError:  global name 'y' is not defined
```

This can raise an error, too

```
sage:  normal_line(1, x^2)
NameError:  global name 'y' is not defined
```

Forgot to define `y`. Fix w/another `try/except` block.

Code & test

```
sage:  def normal_line(a, f, x=x):
         f(x) = f
         df(x) = diff(f)
         try:
           m = 1/df(a)
           try:
             result = y == m*(x - a) + f(a)
           except NameError:
             var('y')
             result = y == m*(x - a) + f(a)
         except ZeroDivisionError:
           result = x == a
         return result
```

Code & test

```
sage:  def normal_line(a, f, x=x):
         f(x) = f
         df(x) = diff(f)
         try:
           m = 1/df(a)
           try:
             result = y == m*(x - a) + f(a)
           except NameError:
             var('y')
             result = y == m*(x - a) + f(a)
         except ZeroDivisionError:
           result = x == a
         return result
sage:  normal_line(1, x^2)
y == 1/2*(x - 1)^2 + 1
```

Should you use exceptions?

- Discouraged in some languages (C++), *but*…
- encouraged in Python/Sage

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- Helps avoid massively nested `if/elif/else` statements, *and*…
- more readable/understandable…
- *especially* if you name expected exception(s)

Should you use exceptions?

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- encouraged in Python/Sage
- Helps avoid massively nested `if/elif/else` statements, *and*…
- more readable/understandable…
- *especially* if you name expected exception(s)
- Relatively small performance penalty

Can raise own exceptions

If need be, you can raise exceptions from your own code:
`raise` *ExceptionType*`,` *string_message*
Possible types:

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`raise` *ExceptionType*`,` *string_message*
Possible types:
Book gives example from finding determinant

Outline

Summary

- Decision making accomplished via `if-elif-else`
  - pseudocode: **if**, **else if**, **else**
- Mathematical examples abound!
  - testing properties of functions
  - piecewise functions
  - finding roots
  - determinants of matrices
- Boolean algebra helps create conditions for `if` and `elif`
  - **and**, **or**, **not**
  - `<=`, `!=`, `>=`
- Sometimes better to catch and handle exceptions