MAT 305: Mathematical Computing

Calculus and Algebra in Sage

John Perry

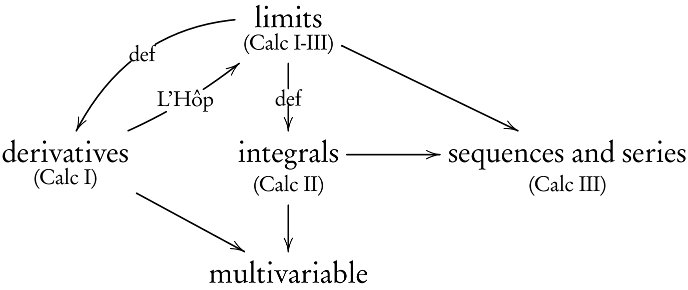
University of Southern Mississippi

Spring 2019

Outline

Outline

Calculus is about *limits*



The `limit()` command

`limit(` $f(x)$, `x=`a, *direction*`)` where

- $f(x)$ is a function in x
- $a \in \mathbb{R}$
- *direction* is optional, but if used has form
  - `dir='left'` or
  - `dir='right'`

Examples

```
sage: limit(x**2-1,x=4)
15

sage: limit(x/abs(x),x=0)
und
```

(Translation: “undefined”)

```
sage: limit(x/abs(x),x=0,dir='right')
1

sage: limit(x/abs(x),x=0,dir='left')
-1

sage: limit(sin(1/x),x=0)
ind
```

(Translation: “indeterminate, but bounded”)

Examples with infinite limits

```
sage: limit(e**(-x),x=infinity)
0

sage: limit((1+1/x)**x,x=infinity)
e
```

(An indeterminate form!)

```
sage: limit((3*x**2-1)/(2*x**2+cos(x)),x=infinity)
3/2

sage: limit(ln(x)/x,x=infinity)
0
```

(Another indeterminate form!)

```
sage: limit(x/ln(x),x=infinity)
+Infinity
```

Careful with infinite limits

```
sage: limit(1/x,x=0)
Infinity
```

Eh, what?

Careful with infinite limits

```
sage: limit(1/x,x=0)
Infinity
```

Eh, what?

> *the limit of the absolute value of the expression is positive infinity, but the limit of the expression itself is not positive infinity or negative infinity*
>
> — *Maxima documentation*

```
sage: limit(1/x,x=0,dir='left')
-Infinity
sage: limit(1/x,x=0,dir='right')
+Infinity
```

Outline

The `diff()` command

`diff(` $f(x)$, x, n`)` where

- $f(x)$ is a function of x
- differentiate f with respect to x
  - *"semi-optional"*: mandatory if f has other variables
  - e.g., partial differentiation, or unknown constants
- *(optional)* compute the nth derivative of $f(x)$

Examples

```
sage: diff(e**x)
e^x
sage: diff(x**10, 5)
30240*x^5
sage: diff(sin(x), 99)
-cos(x)
sage: var('y')
y
sage: diff(x**2+y**2-1)
...output cut...
ValueError: No differentiation variable specified.
sage: diff(x**2+y**2-1, x)
2*x
```

Outline

The `integral()` command

`integral(` $f(x)$, x, $xmin$, $xmax$`)` where

- $f(x)$ is a function of the (optional) variable x
- (optional) $xmin$ and $xmax$ are limits of integration

Example

```
sage: integral(x**2)
1/3*x^3

sage: integral(x**2,x,0,1)
1/3

sage: integral(1/x,x,1,infinity)
...output cut...
ValueError: Integral is divergent.

sage: integral(1/x**2,x,1,infinity)
1
```

Assuming stuff

```
sage: var('p')
p
sage: integral(1/x**p, x, 1, infinity)
ValueError: Computation failed since Maxima
requested additional constraints; using the 'assume'
command before evaluation *may* help (example of
legal syntax is 'assume(q>0)', see 'assume' for more
details)
Is p positive, negative or zero?
```

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legal syntax is 'assume(q>0)', see 'assume' for more
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Is p positive, negative or zero?
sage: assume(p > 1)
sage: integrate(1/x**p, x, 1, infinity)
1/(p - 1)
```

Don’t assume too much (or too little)

```
sage: assume(p <= 1)
ValueError: Assumption is inconsistent
```

Don't assume too much (or too little)

```
sage: assume(p <= 1)
ValueError: Assumption is inconsistent
sage: forget()
sage: assume(p <= 1)
sage: integrate(1/x**p, x, 1, infinity)
ValueError: Computation failed since Maxima
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The problem: $p \leq 0$ implies $\int x^q \, dx$ where $q > 0$

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```
sage: assume(p > 0)
sage: integrate(1/x**p, x, 1, infinity)
ValueError: Integral is divergent.
```

Some other things you can assume

to assume...	type...
$a \neq b$	`assume(a!=b)`
a is positive	`assume(a, 'positive')`
a is an integer	`assume(a, 'integer')`

Some other things you can assume

to assume...	type...
$a \neq b$	`assume(a!=b)`
a is positive	`assume(a, 'positive')`
a is an integer	`assume(a, 'integer')`

When Sage asks about a property, *explore!* If an expression works for `'integer'`, it may work for larger sets, too. Don't be afraid to try `'rational'`, `'real'`, `'complex'` to see what happens.

Numerical integration: Review

Not all integrals can be simplified into elementary functions

Example

$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int e^{-x^2}\, dx$ *(Gaussian error function)*

```
sage: integral(e^(-x^2))
1/2*sqrt(pi)*erf(x)
```

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```

Example

$$\int_{-1}^{1} \sqrt{1 + \frac{4x^2}{1-x^2}}\, dx$$

(arclength of an ellipse)

```
sage: integral(sqrt(1+4*x**2/(1-x**2)),-1,1)
integrate(sqrt(-4*x^2/(x^2 - 1) + 1), x, -1, 1)
```

The `numerical_integral()` command

`numerical_integral(` $f(x)$, *xmin*, *xmax*, *options*`)` where

- $f(x)$ is a function of the defined variable x
- *xmin* and *xmax* are the limits of integration
- *options* include
  - *max_points*, the maximum number of sample points (default: 87)

Gives two results!!!

- approximation to area
- error bound
- returned as Python tuple

Example

```
sage: numerical_integral(sqrt(1+4*x**2/(1-x**2)),
                          -1,1)
(4.8442240644980235, 4.5351915253605327e-06)
```

- error bound is approximately $4.535 \times 10^{-6} \approx .000004535$
- so arclength is approximately $2 \times 4.84422 = 9.68844$

Improving the estimate

```
sage:  numerical_integral(sqrt(1+4*x**2/(1-x**2)),
                                   -1,1,max_points=250)
(4.8442240644980235, 4.5351915253605327e-06)
```

- error bound is approximately $4.535 \times 10^{-6} \approx .000004535$
- so arclength is approximately $2 \times 4.84422 = 9.68844$

Doesn’t seem to improve :-(

Worsening the estimate

```
sage:  numerical_integral(sqrt(1+4*x**2/(1-x**2)),
                          -1,1,max_points=10)
(4.8363135584457568, 0.69875843576683905)
```

- error bound is approximately 0.7…!
- so arclength is somewhere on interval $(4.1, 5.5)$

Ouch!

Accessing only the integral

- `[`$i-1$`]` extracts the ith element of an ordered collection (list, tuple, etc.)
- first entry of result of `numerical_integral()` is the approximation

```
sage: app_int = numerical_integral(
                    sqrt(1+4*x**2/(1-x**2)),-1,1)
sage: app_int[0]
4.8442240644980235
```

Outline

Let's prove a derivative

Recall: $\frac{d}{dx}x^n = nx^{n-1}$

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Anyone remember why?

Let's prove a derivative

Recall: $\frac{d}{dx}x^n = nx^{n-1}$
Anyone remember why?
Definition of the derivative:

$$\frac{d}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

How would we do this in Sage?

You tell me! I've verified that Sage will give us the correct answer.

Do the same for the following

- $\cos x$
- $\arctan x$
- a^x

Outline

Structure

- Mathematical operations take place in well-defined structures
- In this class, we primarily use rings and fields

“Ring”?!? *“Field”?!?*

Ring: ordinary arithmetic guaranteed, *except* division

- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (integers)
- $\mathbb{Q} = \{{}^a/_b : a, b \in \mathbb{Z}, b \neq 0\}$ (rationals, “quotients”)
- $\mathbb{R} = \left\{\pm a_0 a_1 \ldots a_m . a_{m+1} a_{m+1} \ldots\right\}$ (reals, “lengths”)
- $\mathbb{C} = \left\{a + bi : a, b \in \mathbb{R}, i^2 = -1\right\}$ (complex, “complete”)

"Ring"?!? *"Field"?!?*

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Field: division guaranteed, too (except 0)

- $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
- *not* $\mathbb{Z}$
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ not even a ring

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(Intuitive descriptions, not formal definitions)

Sage notation for common rings

- Integers: `ZZ` $\mathbb{Z}$
- Rationals: `QQ` $\mathbb{Q}$
- Reals: `RR` $\mathbb{R}$
  (Sage *approximates* w/53 bits precision)
- Complex: `CC` $\mathbb{C}$
  (Sage *approximates* w/53 bits precision)

Advanced rings

- Algebraic reals: `AA` $\overline{\mathbb{Q}}$
  (algebraic closure of $\mathbb{Q}$)

- Finite fields: `GF`(n) $\mathbb{Z}_n$
  (n power of prime; if not first power, specify string as name for generator)

- Finite rings: `ZZ.quo`(n) $\mathbb{Z}_n$
  (n must be an integer)

- Symbolic: `SR`
  (can use expressions with symbols as entries)

Some complex commands

Suppose $z = a + bi$

`real_part(z)`	a, real part
`imag_part(z)`	b, imaginary part
`norm(z)`	$a^2 + b^2$, Euclidean norm (size)

```
sage: norm(d)
2
sage: norm(d2)
2.00000000000000
```

Modular arithmetic

Remainders v. ring arithmetic

```
sage: Z_23 = ZZ.quo(23)
sage: a = Z_23(5)
sage: a*a
2
sage: a**22
1
```

$\mathbb{Z}_{23}$

$5 \times 5 = 25$, remainder by 23

famous result

Why would you want this?

```
sage: 5**(2**127 - 2) % (2**127 - 1)
RuntimeError: exponent must be at most
9223372036854775807
```

$2^{127}-1$ is prime

$2\times4+3\times6=26$, remainder by 7

Why would you want this?

```
sage: 5**(2**127 - 2) % (2**127 - 1)
RuntimeError: exponent must be at most
9223372036854775807
sage: R = ZZ.quo(2**127 - 1)
sage: a = R(5)
sage: a**(2^127 - 2)
1
```

$2^{127}-1$ is prime

$2\times4+3\times6=26$, remainder by 7

$\mathbb{Z}_{2^{127}-1}$

famous result, again

Sage knows object’s set

```
sage:  5 in R
True
sage:  a in R
True
```

Sage knows object's set

```
sage:  5 in R
True
sage:  a in R
True
sage:  2/3 in R
False
sage:  5.0 in R
False
```

Sage will tell you an object's type

```
sage:  a = R(1)
sage:  type(a)
<type
'sage.rings.finite_rings.integer_mod.IntegerMod_gmp'>
sage:  a = 1
sage:  type(a)
<type 'sage.rings.integer.Integer'>
sage:  b = 2/3
sage:  type(b)
<type 'sage.rings.rational.Rational'>
sage:  d = 1 + I
sage:  type(d)
<type 'sage.symbolic.expression.Expression'>
```

Coercion: move from one type to another

```
sage: d = 1 + I
sage: real_part(d)
sage: type(d)
<type 'sage.symbolic.expression.Expression'>
sage: d2 = CC(1 + I)
sage: type(d2)
<type 'sage.rings.complex_number.ComplexNumber'>
sage: real_part(d)
1
sage: real_part(d2)
1.00000000000000
```

Outline

Question

Can we “divide” in modular arithmetic?

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Ordinary arithmetic: solve $2x = 8$ by division:

$$2x = 8 \quad \Longrightarrow \quad \frac{2x}{2} = \frac{8}{2}$$

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Modular arithmetic: solve $2x = 8$ in $\mathbb{Z}_n$?

Equivalent notion: "Multiplicative inverse"

$$2x = 8 \quad \Longrightarrow \quad \frac{1}{2} \times 2x = \frac{1}{2} \times 8$$

Can we find a "multiplicative inverse" of 2 in $\mathbb{Z}_n$? Let's use Sage with several values of n.

Try $n = 4$ first

```
sage:  R = ZZ.quo(4)
sage:  R(2) * R(0) , R(2) * R(1) , R(2) * R(2) ,
       R(2) * R(3)
```

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sage: R = ZZ.quo(4)
sage: R(2) * R(0) , R(2) * R(1) , R(2) * R(2) ,
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(0, 2, 0, 2)
```

That doesn't look too good. So we can't find a multiplicative inverse for *every* element of *every* modular arithmetic.

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Can we find a multiplicative inverse for *every* element of *some* modular arithmetic?

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That doesn't look too good. So we can't find a multiplicative inverse for *every* element of *every* modular arithmetic.

Can we find a multiplicative inverse for *every* element of *some* modular arithmetic?

(By the way, why did we start at `0` & stop at 4?)

Try $n = 5$ next

```
sage: R = ZZ.quo(5)
sage: R(2) * R(0) , R(2) * R(1) , R(2) * R(2) ,
      R(2) * R(3) , R(2) * R(4)
```

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This time it worked for 2. What about 3 and 4?

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Still $n = 5$

- We found inverses for 2, 3, and 4.

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- `0` is "obviously not, and don't care" (why?)

Still $n = 5$

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- 1 “obviously” has an inverse (why?)
- `0` is “obviously not, and don’t care” (why?)

…so in $\mathbb{Z}_5$ we *can* find multiplicative inverses… for the *nonzero* elements.

Conclusion to example

We found that:

- $\mathbb{Z}_4$ does not have inverses for all elements (not 2 anyway)
- $\mathbb{Z}_5$ has inverses for all nonzero elements

Conclusion to example

We found that:

- $\mathbb{Z}_4$ does not have inverses for all elements (not 2 anyway)
- $\mathbb{Z}_5$ has inverses for all nonzero elements

You can't answer these questions yet, but they should come to mind:

- Why $\mathbb{Z}_5$ and not $\mathbb{Z}_4$?
- What values of n guarantee inverses?
- If not every element has an inverse, *which ones do?*

Outline

Summary

- Sage can do basic calculus & algebra
  - usually works fine
  - may need to supply assumptions
  - bugs can appear; *think* about answers
- Implicit differentiation requires some effort
  - define y as a function of x, not as a variable
- Numerical integration possible
- Ask questions! Experiment! Conjecture! Prove!