

# MAT 305: Mathematical Computing

## Recursion

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Recursion?

The basics

Pascal's triangle

Fibonacci numbers

Issues in  
recursion

Caching

Closed forms (if  
known)

Don't re-curse it, loop  
it!

Eigenbunnies!

Summary

# Outline

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# Recursion?

*re + cursum*: return, travel the path again (Latin)

Two (similar) views:

- **mathematical**: a function defined using itself;
- **computational**: an algorithm that invokes itself.

# When recursion?

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## Summary

- At least one “base” case w/closed form
  - (“closed” = no recursion)
  
- All recursive chains terminate at base case

# Example: Proof by induction

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## Summary

Prove  $P(n)$  for all  $n \in \mathbb{N}$ :

*Inductive Base:* Show  $P(1)$

*Inductive Hypothesis:* Assume  $P(i)$  for  $1 \leq i \leq n$

*Inductive Step:* Show  $P(n+1)$  using  $P(i)$  for  $1 \leq i \leq n$

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# Example: Pascal's triangle

## Recursion?

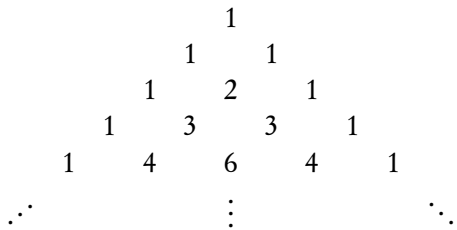
- The basics
- Pascal's triangle**
- Fibonacci numbers

## Issues in recursion

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## Eigenbunnies!

## Summary



# Pascal's triangle $\Rightarrow$ binomial expansion

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## Eigenbunnies!

## Summary

$$\begin{array}{rcccccccc} (x+1)^0 & = & & & & & & & 1 \\ (x+1)^1 & = & & & & & 1x & + & 1 \\ (x+1)^2 & = & & & 1x^2 & + & 2x & + & 1 \\ (x+1)^3 & = & & 1x^3 & + & 3x^2 & + & 3x & + & 1 \\ (x+1)^4 & = & 1x^4 & + & 4x^3 & + & 6x^2 & + & 4x & + & 1 \\ \vdots & & \ddots & & & & & & & & \ddots \end{array}$$

# Do you notice a pattern?

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## Issues in recursion

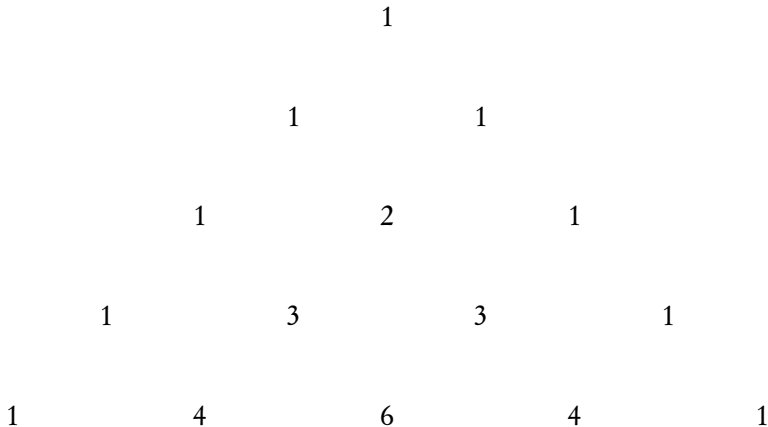
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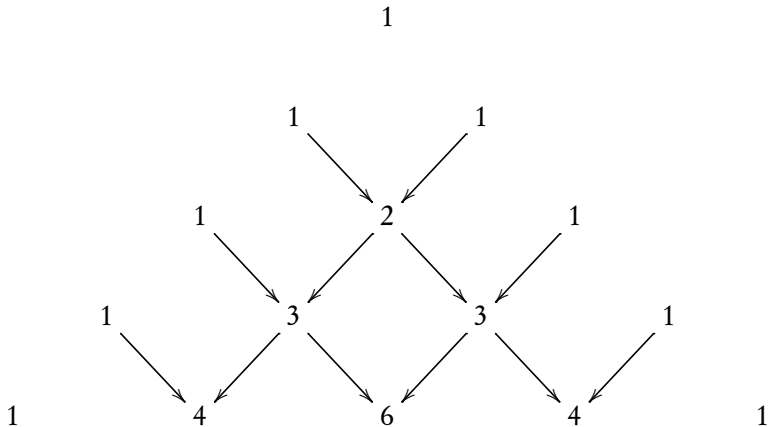
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## Summary



Classic example of recursion.

# Formulating it

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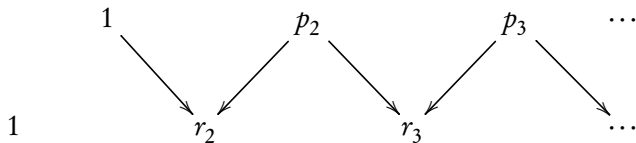
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## Eigenbunnies!

## Summary



$P =$  previous row,  $R =$  current row

- $r_{\text{first}}, r_{\text{last}}$  both 1
- $r_i = p_{i-1} + p_i$

# Pseudocode

**algorithm** *pascals\_row*

**inputs**

- $i \in \mathbb{N}$ , the desired row of Pascal's triangle

**outputs**

- the sequence of numbers in row  $i$  of Pascal's triangle

# Pseudocode

**algorithm** *pascals\_row*

**inputs**

- $i \in \mathbb{N}$ , the desired row of Pascal's triangle

**outputs**

- the sequence of numbers in row  $i$  of Pascal's triangle

**do**

**if**  $i = 1$

$R = [1]$

**else if**  $i = 2$

$R = [1,1]$

# Pseudocode

**algorithm** *pascals\_row*

**inputs**

- $i \in \mathbb{N}$ , the desired row of Pascal's triangle

**outputs**

- the sequence of numbers in row  $i$  of Pascal's triangle

**do**

**if**  $i = 1$

$R = [1]$

**else if**  $i = 2$

$R = [1,1]$

**else**

$P = \text{pascals\_row}(i - 1)$

$R = [1]$

**for**  $j \in (2, 3, \dots, i - 1)$

append  $P_{j-1} + P_j$  to  $R$

append 1 to  $R$

**return**  $R$



## Sage code

```
def pascals_row(i):
    if i == 1:
        R = [1]
    elif i == 2:
        R = [1, 1]
    else:
        # compute previous row first
        P = pascals_row(i - 1)
        # this row starts with 1...
        R = [1]
        # ...adds two above next in this row...
        for j in xrange(1, i - 1):
            R.append(P[j-1] + P[j])
        # ... and ends with 1
        R.append(1)
    return R
```

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# Example

```
sage: pascals_row(3)
```

```
[1, 2, 1]
```

```
sage: pascals_row(5)
```

```
[1, 4, 6, 4, 1]
```

# What happened there?

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The basics

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## Eigenbunnies!

## Summary

```
if i == 1:                                pascals_row(5)
    R = [1]
elif i == 2:
    R = [1, 1]
else:
    P = pascals_row(i - 1)
    R = [1]
    for j in xrange(1, i - 1):
        R.append(P[j-1] + P[j])
    R.append(1)
return R
```

# What happened there?

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if i == 1:
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        R.append(P[j-1] + P[j])
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```

pascals\_row(5)  
pascals\_row(4)

# What happened there?

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```
pascals_row(5)
pascals_row(4)
pascals_row(3)
```

# What happened there?

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pascals\_row(5)  
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# Example: Fibonacci's Bunnies

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## Summary

Fibonacci (Leonardo da Pisa) describes in *Liber Abaci* a population of bunnies:

- first month: one pair of bunnies;

## Example: Fibonacci's Bunnies

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### Summary

Fibonacci (Leonardo da Pisa) describes in *Liber Abaci* a population of bunnies:

- first month: one pair of bunnies;
- second month: pair matures;
- third month: mature pair produces new pair;

## Example: Fibonacci's Bunnies

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### Summary

Fibonacci (Leonardo da Pisa) describes in *Liber Abaci* a population of bunnies:

- first month: one pair of bunnies;
- second month: pair matures;
- third month: mature pair produces new pair;
- fourth month: second pair matures, first pair produces new pair;

## Example: Fibonacci's Bunnies

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### Summary

Fibonacci (Leonardo da Pisa) describes in *Liber Abaci* a population of bunnies:

- first month: one pair of bunnies;
- second month: pair matures;
- third month: mature pair produces new pair;
- fourth month: second pair matures, first pair produces new pair;
- fifth month: third pair matures, two mature pairs produce new pairs;
- ...

# How many pairs?

<b>month</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>...</b>
<b>mature pairs</b>								
<b>immature pairs</b>								
<b>new pairs</b>	<b>1</b>							
<b>total pairs</b>	<b>1</b>							

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## Eigenbunnies!

## Summary

# How many pairs?

<b>month</b>	1	2	3	4	5	6	7	...
<b>mature pairs</b>								
<b>immature pairs</b>		1						
<b>new pairs</b>	1							
<b>total pairs</b>	1	1						

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Summary

# How many pairs?

month	1	2	3	4	5	6	7	...
mature pairs			1					
immature pairs		1						
new pairs	1		1					
total pairs	1	1	2					

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## Summary



# How many pairs?

month	1	2	3	4	5	6	7	...
mature pairs			1	1				
immature pairs		1		1				
new pairs	1		1	1				
total pairs	1	1	2	3				

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## Summary

# How many pairs?

month	1	2	3	4	5	6	7	...
mature pairs			1	1	2			
immature pairs		1		1	1			
new pairs	1		1	1	2			
total pairs	1	1	2	3	5			

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## Summary

# How many pairs?

<b>month</b>	1	2	3	4	5	6	7	...
<b>mature pairs</b>			1	1	2	3		
<b>immature pairs</b>		1		1	1	2		
<b>new pairs</b>	1		1	1	2	3		
<b>total pairs</b>	1	1	2	3	5	8		

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Summary

# How many pairs?

<b>month</b>	1	2	3	4	5	6	7	...
<b>mature pairs</b>			1	1	2	3	5	
<b>immature pairs</b>		1		1	1	2	3	
<b>new pairs</b>	1		1	1	2	3	5	
<b>total pairs</b>	1	1	2	3	5	8	13	...

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## Summary

# Describing it

<b>month</b>	1	2	3	4	5	6	7	...
<b>mature pairs</b>			1	1	2	3	5	
<b>immature pairs</b>		1		1	1	2	3	
<b>new pairs</b>	1		1	1	2	3	5	
<b>total pairs</b>	1	1	2	3	5	8	13	...

- $\text{total} = (\# \text{ mature} + \# \text{ immature}) + \# \text{ new}$

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## Eigenbunnies!

## Summary

<b>month</b>	1	2	3	4	5	6	7	...
<b>mature pairs</b>			1	1	2	3	5	
<b>immature pairs</b>		1		1	1	2	3	
<b>new pairs</b>	1		1	1	2	3	5	
<b>total pairs</b>	1	1	2	3	5	8	13	...

- $\text{total} = (\# \text{ mature} + \# \text{ immature}) + \# \text{ new}$
- $\text{total} = \# \text{ one month ago} + \# \text{ new}$

# Describing it

<b>month</b>	1	2	3	4	5	6	7	...
<b>mature pairs</b>			1	1	2	3	5	
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- $\text{total} = \# \text{ one month ago} + \# \text{ mature now}$

## Recursion?

The basics

Pascal's triangle

Fibonacci numbers

## Issues in recursion

Caching

Closed forms (if  
known)

Don't re-curse it, loop  
it!

## Eigenbunnies!

## Summary

# Describing it

<b>month</b>	1	2	3	4	5	6	7	...
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- $\text{total} = \# \text{ one month ago} + \# \text{ two months ago}$



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- $\text{total} = \# \text{ one month ago} + \# \text{ two months ago}$

$$\therefore F_{\text{now}} = F_{\text{one month ago}} + F_{\text{two months ago}}, \text{ or}$$

# Describing it

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Summary

<b>month</b>	1	2	3	4	5	6	7	...
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<b>total pairs</b>	1	1	2	3	5	8	13	...

- total = (# mature + # immature) + # new
- total = # one month ago + # new
- total = # one month ago + # mature now
- total = # one month ago + # two months ago

$$\therefore F_{\text{now}} = F_{\text{one month ago}} + F_{\text{two months ago}}, \text{ or}$$

$$F_i = F_{i-1} + F_{i-2}$$

## ∴ Fibonacci Sequence

$$F_i = \begin{cases} 1, & i = 1, 2; \\ F_{i-1} + F_{i-2}, & i \geq 3. \end{cases}$$

## ∴ Fibonacci Sequence

$$F_i = \begin{cases} 1, & i = 1, 2; \\ F_{i-1} + F_{i-2}, & i \geq 3. \end{cases}$$

### Example

$$\begin{aligned} F_5 &= F_4 + F_3 \\ &= (F_3 + F_2) + (F_2 + F_1) \\ &= [(F_2 + F_1) + F_2] + (F_2 + F_1) \\ &= 3F_2 + 2F_1 \\ &= 5. \end{aligned}$$

## ∴ Fibonacci Sequence

$$F_i = \begin{cases} 1, & i = 1, 2; \\ F_{i-1} + F_{i-2}, & i \geq 3. \end{cases}$$

### Example

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$$\begin{aligned} F_{100} &= F_{99} + F_{98} \\ &= \dots \\ &= 218922995834555169026 \cdot F_2 + 135301852344706746049 \cdot F_1 \\ &= 354224848179261915075 \end{aligned}$$

# Pseudocode

Easy to implement w/recursion:

**algorithm** *Fibonacci*

**inputs**

$$n \in \mathbb{N}$$

**outputs**

the  $n$ th Fibonacci number

**do**

**if**  $n = 1$  **or**  $n = 2$

**return** 1

**else**

**return**  $Fibonacci(n - 2) + Fibonacci(n - 1)$

Recursion?

The basics

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it!

Eigenbunnies!

Summary

# Implementation

```
sage: def fibonacci(n):  
      if n == 1 or n == 2:  
          return 1  
      else:  
          return fibonacci(n-2) + fibonacci(n-1)
```

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Eigenbunnies!

Summary

# Implementation

```
sage: def fibonacci(n):  
        if n == 1 or n == 2:  
            return 1  
        else:  
            return fibonacci(n-2) + fibonacci(n-1)
```

```
sage: fibonacci(5)  
5
```

```
sage: fibonacci(20)  
6765
```

```
sage: fibonacci(30)  
832040
```



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## ③ Eigenbunnies!

## ④ Summary

# Issues in recursion

- Infinite loops
  - recursion must stop eventually
  - must ensure reach base case

# Issues in recursion

- Infinite loops
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  - `fibonacci(20)` requires `fibonacci(19)` and `fibonacci(18)`
  - `fibonacci(19)` *also* requires `fibonacci(18)`
  - $\therefore$  `fibonacci(18)` computed twice!

# Issues in recursion

- Infinite loops
  - recursion must stop eventually
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  - `fibonacci(20)` requires `fibonacci(19)` and `fibonacci(18)`
  - `fibonacci(19)` *also* requires `fibonacci(18)`
  - $\therefore$  `fibonacci(18)` computed twice!
- Limit to recursion
  - `pascals_row(1000)`

## Example

Modify program:

```
sage: def fibonacci_details(n):  
        print 'computing fibonacci #', n,  
        if n == 1 or n == 2:  
            return 1  
        else:  
            return fibonacci_details(n-2)  
                + fibonacci_details(n-1)
```

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it!

Eigenbunnies!

Summary

## Example

Modify program:

```
sage: def fibonacci_details(n):
        print 'computing fibonacci #', n,
        if n == 1 or n == 2:
            return 1
        else:
            return fibonacci_details(n-2)
                + fibonacci_details(n-1)
```

```
sage: fibonacci_details(5)
computing fibonacci # 5 computing fibonacci # 3
computing fibonacci # 1 computing fibonacci # 2
computing fibonacci # 4 computing fibonacci # 2
computing fibonacci # 3 computing fibonacci # 1
computing fibonacci # 2
5
```

## Example

Modify program:

```
sage: def fibonacci_details(n):
        print 'computing fibonacci #', n,
        if n == 1 or n == 2:
            return 1
        else:
            return fibonacci_details(n-2)
                + fibonacci_details(n-1)
```

```
sage: fibonacci_details(5)
computing fibonacci # 5 computing fibonacci # 3
computing fibonacci # 1 computing fibonacci # 2
computing fibonacci # 4 computing fibonacci # 2
computing fibonacci # 3 computing fibonacci # 1
computing fibonacci # 2
5
```

... $F_3$  computed 2 times;  $F_2$ , 3 times;  $F_1$ , 2 times

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## ③ Eigenbunnies!

## ④ Summary



# Workaround

Can we tell Sage to “remember” pre-computed values?

- Need a list
- Compute  $F_i$ ? add value to list
- Apply formula *only* if  $F_i$  not in list!
- “Remember” computation after function ends: **global** list
  - (called a **cache**)

## Recursion?

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it!

## Eigenbunnies!

## Summary

Can we tell Sage to “remember” pre-computed values?

- Need a list
- Compute  $F_i$ ? add value to list
- Apply formula *only* if  $F_i$  not in list!
- “Remember” computation after function ends: **global** list
  - (called a **cache**)

## Definition

- **global** variables available to all functions in system
- **cache** makes information quickly accessible

Recursion?

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known)

Don't re-curse it, loop  
it!

Eigenbunnies!

Summary

**algorithm** *Fibonacci\_with\_table*

**globals**  $F$ , a list of integers, initially  $[1, 1]$

**inputs**

$n \in \mathbb{N}$

**outputs**

the  $n$ th Fibonacci number

**do**

**if**  $n > \#F$

Let  $a = \text{Fibonacci\_with\_table}(n - 1)$

Let  $b = \text{Fibonacci\_with\_table}(n - 2)$

Append  $a + b$  to  $F$

**return**  $F_n$

# Hand implementation

```
sage: F = [1,1]
```

```
sage: def fibonacci_with_table(n):  
    global F  
    if n > len(F):  
        print 'computing fibonacci #', n,  
        a = fibonacci_with_table(n-2)  
        b = fibonacci_with_table(n-1)  
        F.append(a + b)  
    return F[n-1]
```

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Summary

# Hand implementation

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Don't re-curse it, loop  
it!

Eigenbunnies!

Summary

```
sage: F = [1,1]
sage: def fibonacci_with_table(n):
        global F
        if n > len(F):
            print 'computing fibonacci #', n,
            a = fibonacci_with_table(n-2)
            b = fibonacci_with_table(n-1)
            F.append(a + b)
        return F[n-1]
```

## Example

```
sage: fibonacci_with_table(5)
computing fibonacci # 5 computing fibonacci # 4
computing fibonacci # 3
5
```

# But... no need to implement!

```
sage: @cached_function
def fibonacci_cached(n):
    print 'computing fibonacci #', n,
    if n == 1 or n == 2:
        return 1
    else:
        return fibonacci_cached(n-2)
        + fibonacci_cached(n-1)
```

Recursion?

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known)

Don't re-curse it, loop  
it!

Eigenbunnies!

Summary

## But... no need to implement!

```
sage: @cached_function
def fibonacci_cached(n):
    print 'computing fibonacci #', n,
    if n == 1 or n == 2:
        return 1
    else:
        return fibonacci_cached(n-2)
        + fibonacci_cached(n-1)
```

### Example

```
sage: fibonacci(5)
computing fibonacci # 5 computing fibonacci # 3
computing fibonacci # 1 computing fibonacci # 2
computing fibonacci # 4
5
```

Recursion?

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## ④ Summary



However...

Avoid recursion when possible

- can often rewrite as a loop
- can sometimes rewrite in “closed form”

However...

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- can often rewrite as a loop
- can sometimes rewrite in “closed form”

Example

“Closed form” for Fibonacci sequence:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

However...

Avoid recursion when possible

- can often rewrite as a loop
- can sometimes rewrite in “closed form”

Example

“Closed form” for Fibonacci sequence:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Coincidence? I think not...

$$\frac{1 + \sqrt{5}}{2} = \textit{golden ratio}$$

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Don't re-curse it, loop it!

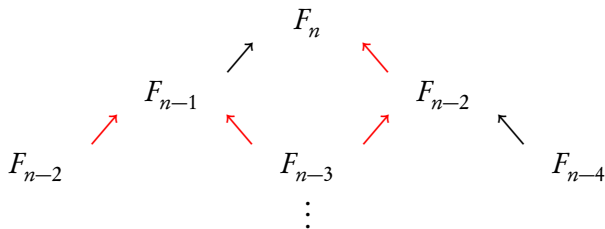
## ③ Eigenbunnies!

## ④ Summary

# Looped Fibonacci: How?

We will *not* use the closed form, but a loop

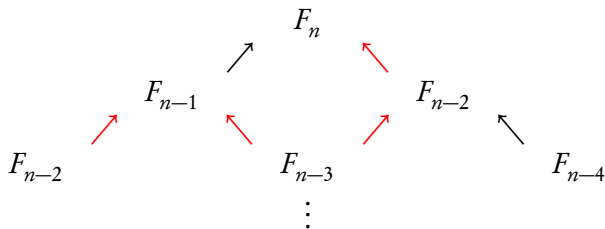
- Recursive: down, then up, then down, then up...



# Looped Fibonacci: How?

We will *not* use the closed form, but a loop

- Recursive: down, then up, then down, then up...



- Looped: only up, directly!
  - $F_2 \xrightarrow{+F_1} F_3 \xrightarrow{+F_2} \dots \xrightarrow{+F_{n-2}} F_n$
  - remember two previous computations
  - remember?  $\implies$  variables

# Looped Fibonacci: Pseudocode

**algorithm** *Looped\_Fibonacci*

**inputs**

$$n \in \mathbb{N}$$

**outputs**

the  $n$ th Fibonacci number

**do**

– Define the base case

Let  $F_{\text{prev}} = 1, F_{\text{curr}} = 1$

– Use the formula to move forward to  $F_n$

**for**  $i \in \{3, \dots, n\}$

– Compute next element, then move forward

Let  $F_{\text{next}} = F_{\text{prev}} + F_{\text{curr}}$

Let  $F_{\text{prev}} = F_{\text{curr}}$

Let  $F_{\text{curr}} = F_{\text{next}}$

**return**  $F_{\text{curr}}$

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known)

Don't re-recurse it, loop  
it!

Eigenbunnies!

Summary

# Looped Fibonacci: Implementation

```
sage: def looped_Fibonacci(n):  
      Fprev = 1  
      Fcurr = 1  
      for i in xrange(3,n+1):  
          Fnext = Fprev + Fcurr  
          Fprev = Fcurr  
          Fcurr = Fnext  
      return Fcurr
```



# Looped Fibonacci: Implementation

```
sage: def looped_Fibonacci(n):  
      Fprev = 1  
      Fcurr = 1  
      for i in xrange(3,n+1):  
          Fnext = Fprev + Fcurr  
          Fprev = Fcurr  
          Fcurr = Fnext  
      return Fcurr
```

```
sage: looped_Fibonacci(100)  
354224848179261915075
```

*(Much faster than recursive version)*

# Faster, too

## Recursion?

The basics

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Fibonacci numbers

## Issues in recursion

Caching

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known)

Don't re-curse it, loop  
it!

## Eigenbunnies!

## Summary

```
sage: %time a = looped_Fibonacci(30000)
```

```
CPU time: 0.01 s, Wall time: 0.01 s
```

```
sage: %time a = Fibonacci_with_table(30000)
```

```
CPU time: probably crashes, Wall time: if not, get  
some coffee
```

```
sage: %time a = Fibonacci(10000)
```

```
CPU time: probably crashes, Wall time: if not,  
come back tomorrow
```

# Recursive vs. Looped vs. Closed-form

- Recursive

**pros:** simple to write, “naïve” approach

**cons:** slower, memory intensive, *indefinite loop*  
*w/out loop structure*

# Recursive vs. Looped vs. Closed-form

- Recursive

**pros:** simple to write, “naïve” approach

**cons:** slower, memory intensive, *indefinite loop*  
*w/out loop structure*

- Looped (also called **dynamic programming**)

**pros:** not too slow, not too complicated, loop can  
be definite

**cons:** not as simple as recursive, sometime not  
obvious

# Recursive vs. Looped vs. Closed-form

- **Recursive**
  - pros:** simple to write, “naïve” approach
  - cons:** slower, memory intensive, *indefinite loop w/out loop structure*
- **Looped (also called **dynamic programming**)**
  - pros:** not too slow, not too complicated, loop can be definite
  - cons:** not as simple as recursive, sometime not obvious
- **Closed-form**
  - pros:** one step (no loop)
  - cons:** finding it often requires *significant* effort

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## ④ Summary

# Neat fact of eigenvectors

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Don't re-curse it, loop  
it!

Eigenbunnies!

Summary

## Theorem (Eigendecomposition)

Let  $M$  be an  $n \times n$  matrix with

- independent eigenvectors  $\mathbf{e}_1, \dots, \mathbf{e}_n$
- corresponding to eigenvalues  $\lambda_1, \dots, \lambda_n$ .

We can rewrite  $M$  as  $M = Q\Lambda Q^{-1}$  where

$$Q = (\mathbf{e}_1 | \mathbf{e}_2 | \dots | \mathbf{e}_n) \quad \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

## Example

With  $M$  as defined,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 2 & \\ & -2 \end{pmatrix}$$

Verify in Sage that  $M = Q\Lambda Q^{-1}$



## Example

With  $M$  as defined,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 2 & \\ & -2 \end{pmatrix}$$

Verify in Sage that  $M = Q\Lambda Q^{-1}$

```
sage: Q = matrix(2,2,[1,1,1,-1])
```

```
sage: L = matrix(2,2,[2,0,0,-2])
```

```
sage: Q*L*Q**(-1)
```

```
[0 2]
```

```
[2 0]
```

... recall  $M = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

# But how is this useful?

## Recursion?

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Don't re-curse it, loop  
it!

## Eigenbunnies!

## Summary

Consider the numbers

1, 1, 2, 3, 5, 8, 13, ...

## But how is this useful?

Consider the numbers

1, 1, 2, 3, 5, 8, 13, ...

This is the well-known Fibonacci sequence:

$$f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2}$$

Can we get a “non-recursive” formula?

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Eigenbunnies!

Summary

## Fibonacci matrix

As a matrix equation,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_{n-2} \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$$

Let's try rewriting the matrix

$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

## Fibonacci matrix

As a matrix equation,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_{n-2} \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$$

Let's try rewriting the matrix

$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Iterative multiplication generates the sequence

```
sage: F = matrix(2,2,[1,1,1,0])
```

```
sage: f12 = vector([1,1])
```

```
sage: F*f12
```

```
[2, 1]
```

```
sage: F^2*f12
```

```
[3, 2]
```

```
sage: F^3*f12
```

```
[5, 3]
```

$$F^{n-2} \begin{pmatrix} f_2 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$$

and

$$\begin{aligned} F^{n-2} &= (Q\Lambda Q^{-1})^{n-2} \\ &= \underbrace{(Q\Lambda Q^{-1})(Q\Lambda Q^{-1})\cdots(Q\Lambda Q^{-1})}_{n-2} \\ &= \underbrace{Q\Lambda(Q^{-1}Q)\Lambda(Q^{-1}Q)\cdots(Q^{-1}Q)\Lambda Q^{-1}}_{n-2} \\ &= Q\Lambda^{n-2}Q^{-1} \end{aligned}$$

Since  $\Lambda$  is diagonal, it is easy to compute  $\Lambda^n$

Recursion?

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Closed forms (if  
known)

Don't re-curse it, loop  
it!

Eigenbunnies!

Summary

# What to do?

## General outline:

- Compute eigenvectors and eigenvalues  
`sage: F.eigenvectors_right()`
- Construct  $Q\Lambda^n Q^{-1}$   
`sage: Q = matrix(2,2,[...])`  
`sage: L = matrix(2,2,[...])`
- Analyze the equation

# One “drawback”

- eigenvectors, eigenvalues look inexact

```
sage: F.eigenvectors_right()  
[(-0.618033988749895?,  
  [(1, -1.618033988749895?)], 1),  
 (1.618033988749895?,  
  [(1, 0.618033988749895?)], 1)]
```

## Recursion?

The basics

Pascal's triangle

Fibonacci numbers

## Issues in recursion

Caching

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## One “drawback”

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- eigenvectors, eigenvalues look inexact

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  [(1, 0.618033988749895?)], 1)]
```

- In fact, we can determine their exact values

```
sage: edata = F.eigenvectors_right()  
sage: lam1, lam2 = edata[0][0], edata[1][0]  
sage: lam1 = lam1.radical_expression(); lam1  
-1/2*sqrt(5) + 1/2  
sage: lam2 = lam2.radical_expression(); lam2  
1/2*sqrt(5) + 1/2
```

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# Put it together

```
[(-1/2*sqrt(5) + 1/2, [(1, -1/2*sqrt(5) - 1/2)], 1),  
 (1/2*sqrt(5) + 1/2, [(1, 1/2*sqrt(5) - 1/2)], 1)]
```

```
sage: Q = matrix(  
        [1, -1/2*sqrt(5) - 1/2],  
        [1, 1/2*sqrt(5) - 1/2]  
    )  
sage: var('n')  
sage: L = matrix(2,2,[  
        (-1/2*sqrt(5) + 1/2)^(n-2), 0,  
        0, (1/2*sqrt(5) + 1/2)^(n-2)  
    ])  
sage: Q*L*Q**(-1)  
...very unpleasant
```

Recursion?

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Eigenbunnies!

Summary

Let  $M = (Q\Lambda^n Q^{-1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , and let  $f_n = M_{1,1}$  (the top entry).

An “algebraic massage” (`.full_simplify()`) gives

$$f_n = \frac{\sqrt{5}}{10} \left[ (3 + \sqrt{5}) \left( \frac{1 + \sqrt{5}}{2} \right)^{n-2} - (3 - \sqrt{5}) \left( \frac{1 - \sqrt{5}}{2} \right)^{n-2} \right],$$

already a “pleasant” closed form, and thus what we wanted.

...or is it?

Recursion?

- The basics
- Pascal's triangle
- Fibonacci numbers

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recursion

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Summary

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already a “pleasant” closed form, and thus what we wanted.

But we can do better!

## More algebraic massage...

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### Eigenbunnies!

### Summary

Use Sage (in particular, `expand()`) to verify that

$$3 + \sqrt{5} = 2 \left( \frac{1 + \sqrt{5}}{2} \right)^2 \quad \text{and} \quad 3 - \sqrt{5} = 2 \left( \frac{1 - \sqrt{5}}{2} \right)^2 .$$

## More algebraic massage...

Use Sage (in particular, `expand()`) to verify that

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We can use this fact to rewrite

$$f_n = \frac{\sqrt{5}}{10} \left[ (3 + \sqrt{5}) \left( \frac{1 + \sqrt{5}}{2} \right)^{n-2} - (3 - \sqrt{5}) \left( \frac{1 - \sqrt{5}}{2} \right)^{n-2} \right]$$

as...

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Summary

# Binet's Formula

$$f_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$

golden ratio

(kindly observe a moment of reverent awe)

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Summary

# Outline

## ① Recursion?

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## ② Issues in recursion

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## ③ Eigenbunnies!

## ④ Summary



# Summary

- Recursion: function defined using other values of function
- Issues
  - can waste computation
  - can lead to infinite loops (bad design)
- Use when
  - closed/loop form too complicated
  - chains not too long
  - “memory table” feasible