MAT 305: Mathematical Computing

Looping with indefinite loops

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Outline

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Loops?

- **loop:** a sequence of statements that is repeated

big time bug: *infinite loops*

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big time bug: *infinite loops*

"infinite loop"?

see infinite loop

— *AmigaDOS* manual, ca. 1993

Why loops?

- like functions: avoid retyping code
  - many patterns repeated
  - same behavior, different data
- don't know number of repetitions when programming

Types of loops

- definite
  - # repetitions known at outuset
- indefinite
  - # repetitions not known / unknowable at outset

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Most languages use different constructions for each

Outline

The `while` command

`while` *condition* `:`
 statement1
 statement2
 ...
where

- statements are executed while *condition* remains true
  - statements will *not* be executed if *condition* is false from the get-go
- like definite loops, variables in *condition* can be modified
- unlike definite loops, variables in *condition* **should** be modified

Pseudocode for indefinite loop

while *condition*
    statement1
    statement2
    ...
out-of-loop statement 1

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    statement2
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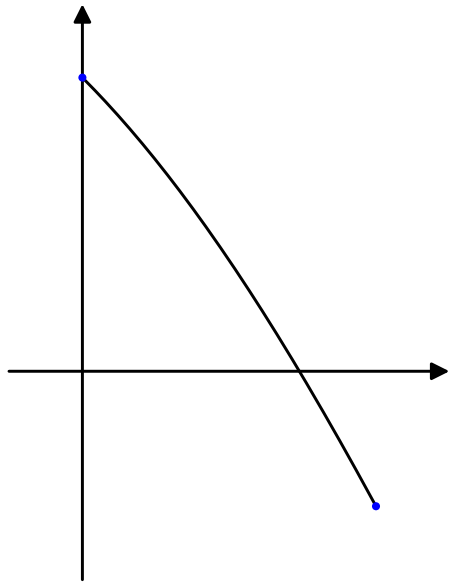
Notice:

- indentation ends at end of loop
- no colon

Example

```
sage:  f = x**10
sage:  while f != 0:
           f = diff(f)
           print f
10*x^9
90*x^8
720*x^7
5040*x^6
30240*x^5
151200*x^4
604800*x^3
1814400*x^2
3628800*x
3628800
0
```

Bisection again!



(need Acrobat Reader to see animation)

This time let's prolong the loop until d digits agree

Pseudocode

algorithm *method_of_bisection*

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inputs

f, a continuous function

$a, b \in \mathbb{R}$ such that $a \neq b$ **and** $f(a)$ and $f(b)$ have different signs

d, a positive integer

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outputs

$c \in [a, b]$ such that $f(c) \approx 0$ **and** c accurate to d digits

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do

while the digits of a and b differ through d digits

Let $c = \frac{a+b}{2}$

if $f(a)$ **and** $f(c)$ have the same sign

Let $a = c$ Interval now $\left(\frac{a+b}{2}, b\right)$

else if $f(a)$ and $f(c)$ have opposite signs

Let $b = c$ Interval now $\left(a, \frac{a+b}{2}\right)$

else

we must have $f(c) = 0$

return c

return a, rounded to hundredths place

How to check digits?

Round, of course!

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Round, of course! . . . Oh, really? How far should we round?

Example (Do π and 3.141 agree on first two three digits?)

```
sage:  round(pi, 3) == round(3.141, 3)
false
```

How to check digits?

Round, of course! . . . Oh, really? How far should we round?

Example (Do π and 3.141 agree on first two three digits?)

```
sage:  round(pi, 3) == round(3.141, 3)
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```

Think about it: $\pi \approx 3.1415$ rounds to 3.142

Truncate, instead

- Multiply by power of 10
- Convert to integer
- Convert back to float
- Divide by power of 10

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$$3.14159 \longrightarrow 3141.59 \longrightarrow 3141 \longrightarrow 3.141$$

Sage code to do this

```
sage:  def trunc(a, d=2):
           a *= 10^d
           a = int(a)
           a = float(a)
           return a/10^d
```

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(3.141, True)
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(3.141, True)
```

Maybe add this to your `calc_utils.sage` script?

(kind of amazed this isn't built in)

Try it!

```
sage:  def method_of_bisection(f, a, b, d=2, x=x):
         f(x) = f
         while trunc(a, d) != trunc(b, d):
```

Try it!

```
sage:  def method_of_bisection(f, a, b, d=2, x=x):
         f(x) = f
         while trunc(a, d) != trunc(b, d):
           c = (a + b)/2
```

Try it!

```
sage:  def method_of_bisection(f, a, b, d=2, x=x):
         f(x) = f
         while trunc(a, d) != trunc(b, d):
           c = (a + b)/2
           if f(a)*f(c) > 0:
             a = c
           elif f(a)*f(x) < 0:
             b = c
           else:
             return c
         return round(a,d)
```

Try it!

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sage:  def method_of_bisection(f, a, b, d=2, x=x):
         f(x) = f
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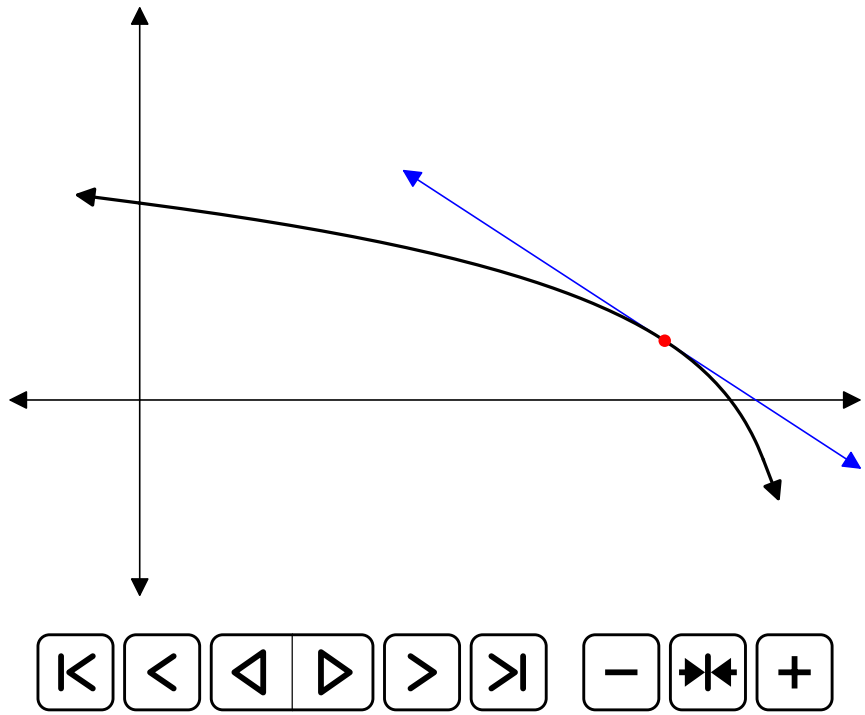
sage:  method_of_bisection(cos(x)-x,x,0,1)
0.74
```

Outline

Another way of finding a root

- tangent line approximates f
- start close to root? line's root should approximate f's root
- repeat as long as first d digits change



Problem analysis

We need to:

- compute tangent line
- find line's root
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How do we decide if first d digits changed?

- `trunc()` again!
- compare current, previous approx's
- *need to remember previous!*

Pseudocode

algorithm *newtons_method*
inputs
  f, a differentiable function
  a, approximation to a root of f
  d, positive number
outputs
  b, "better" approximation to a root of f
do
  let $b = a$
  let $a = b - 1$
  while a, b differ in first d digits
    let $a = b$
    let b be root of line tangent to f at $x = a$
  return b

What are this line and this one up to?

why?

Sage code

Avoid re-inventing the wheel: re-attach `calc_utils.sage`

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```
sage:  def newtons_method(f, a, b, d=2, x=x):
           f(x) = f
           df(x) = diff(f, x)
           b, a = a, a - 1
           while trunc(a, d) != trunc(b, d):
             a = b
             sols = solve(tangent_line(f, a, x), x)
             b = sols[0].rhs()
           return b
```

Sage code

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```

works great, except:

```
sage:  newtons_method(cos(x) - x, 0.5, 4)
1/75485362136393*(75485362136393*cos(57008237648741/754
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Sage solves for the line's root exactly!

Sage code

How can we get around it?

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sage:  def newtons_method(f, a, b, d=2, x=x):
           f(x) = f
           df(x) = diff(f, x)
           b, a = a, a - 1
           while trunc(a, d) != trunc(b, d):
             a = b
             sols = solve(tangent_line(f, a, x), x)
             b = float(sols[0].rhs())
           return b
sage:  newtons_method(cos(x) - x, 0.5, 4)
0.7390851332151602
```

Outline

Gaussian integers

Definition

A **Gaussian integer** has the form $a + bi$ where $a, b \in \mathbb{Z}$ and $i^2 = -1$.

Examples

$$7, \quad 2 + 3i, \quad -3 + 2i$$

but *definitely not*

$$\frac{3}{2}, \quad \pi, \quad \frac{1}{3} - i\frac{5}{2}.$$

Gaussian integers form a ring

$$(a+bi) \pm (c+di) = (a \pm c) + i(b \pm d)$$

$$(a+bi)(c+di) = (ac-bd) + i(ad+bc)$$

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Can we also *divide* by Gaussian integers?

Analyze the problem

What does division mean, anyway?

Analyze the problem

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Repeated subtraction.

Example

$$40 = 13 \times 3 + 1$$

Extend the meaning

Integer division:

Subtract until the remainder's size is less than the divisors's.

Gaussian integer division:

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Integer division:

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Gaussian integer division:

Subtract until the remainder's size is less than the divisor's.

What do we mean by "remainder's size"?

Size of a number

In $\mathbb{Z}$, "size" is called **absolute value**:

$$|-5| = |5| = 5$$

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In $\mathbb{Z}[i]$, "size" is called **(Euclidean) norm**:

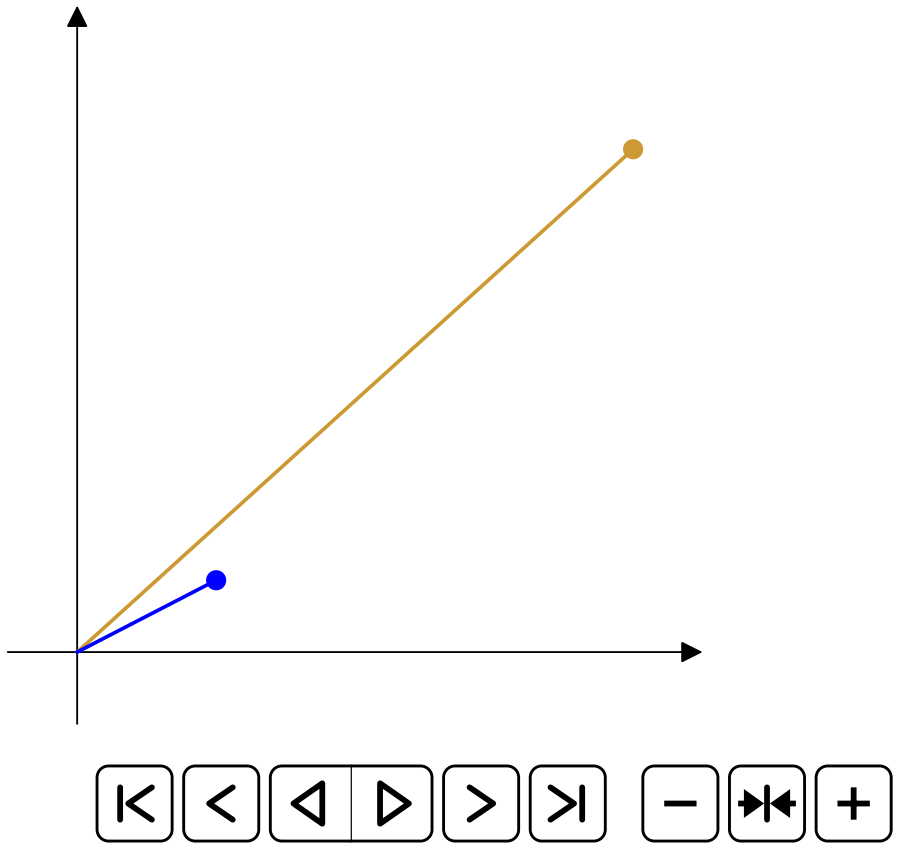
$$||a+bi|| = a^2 + b^2$$

Example

$||2+3i|| = 2^2 + 3^2 = 13$

Geometrically

Divide $8+7i$ by $2+i$:



A wrench in the system…

We found that

$$8+7i = 5 \times (2+i) + (-2+2i)$$

but…

A wrench in the system...

We found that

$$8+7i = 5 \times (2+i) + (-2+2i)$$

but...

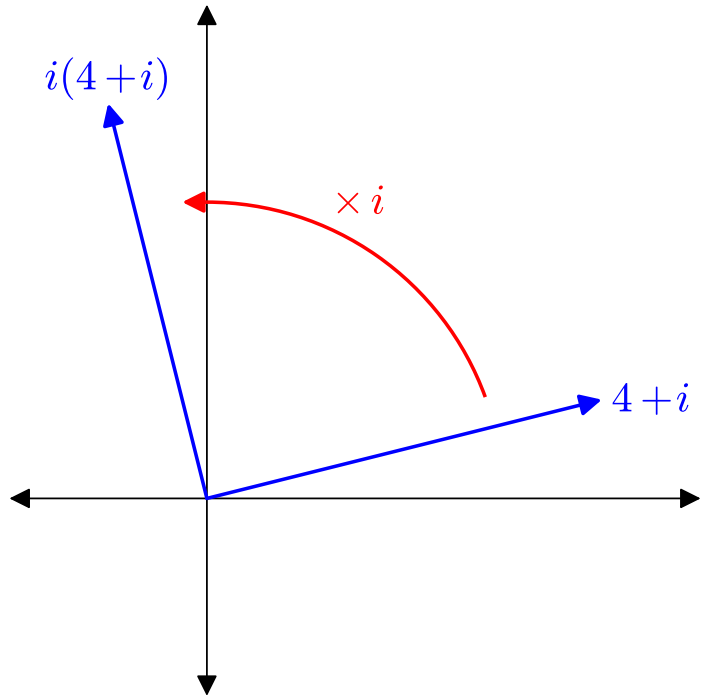
$$||-2+2i|| = 4+4 \quad > \quad 4+1 = ||2+i||$$

The distance is larger than we'd like!

Can we do better?

Recall Programming #4 in "Pretty Pictures":

Multiplying a complex number by i rotates it 90° counterclockwise.



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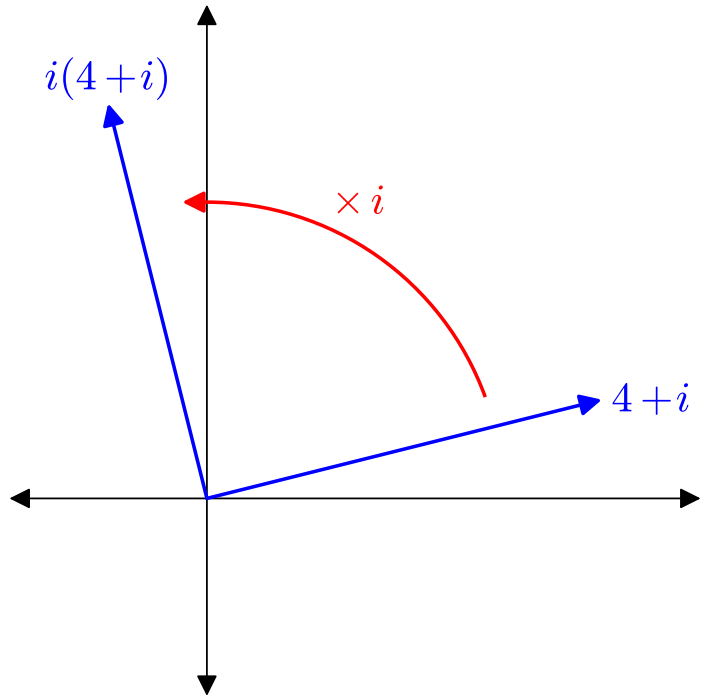
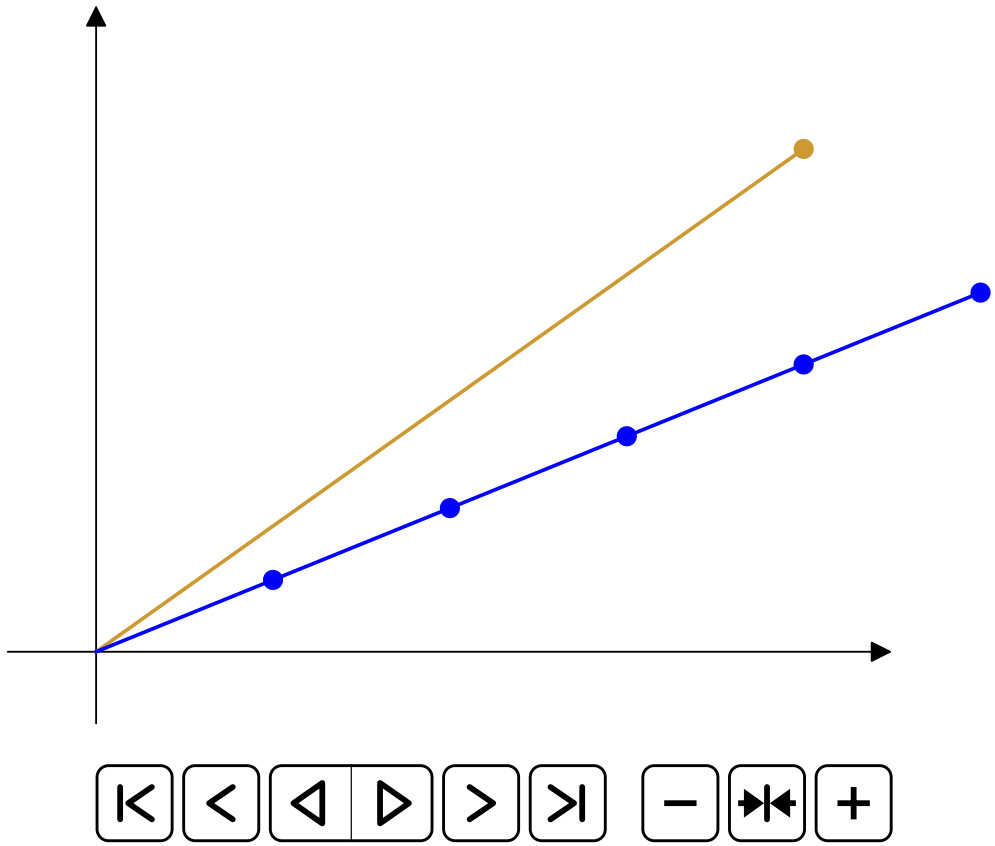


…so we *should* get closer if we add *imaginary* multiples of $2+i$.

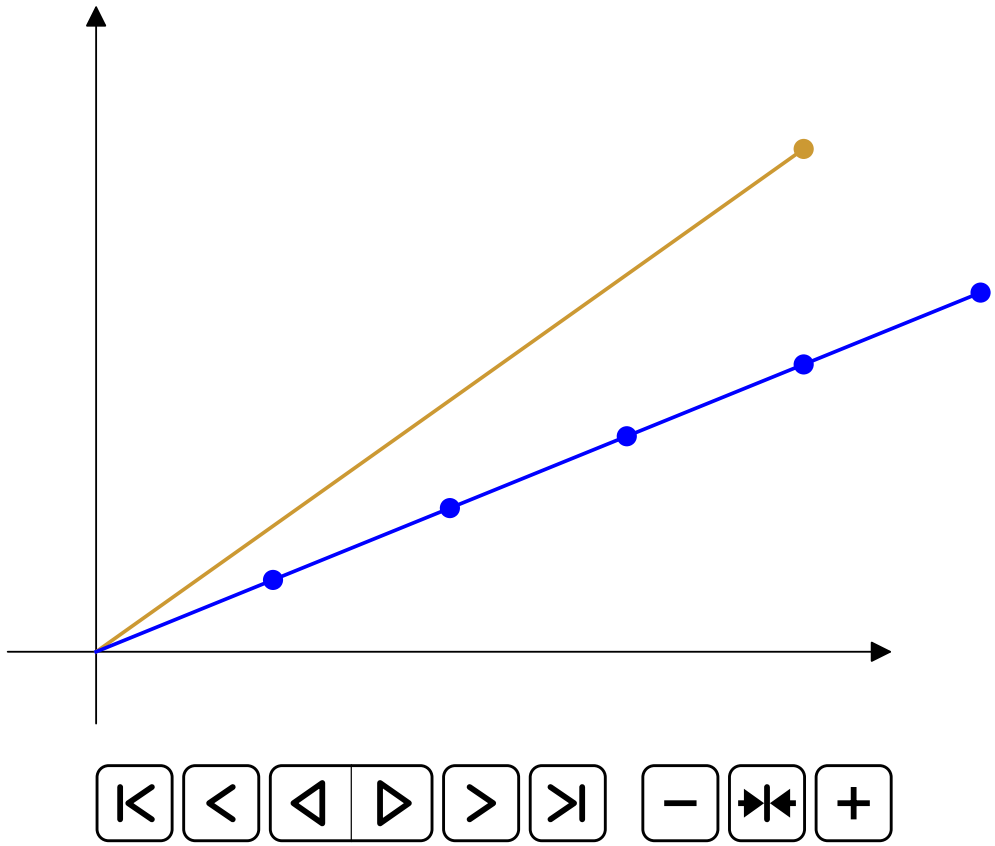
Geometrically (again)

Divide $8+7i$ by $2+i$ *completely*:



Geometrically (again)

Divide $8+7i$ by $2+i$ *completely*:



We have $8+7i = (5+i)(2+i)+(-1)$

Pseudocode

algorithm *gaussian_reduction*
inputs
$z, d \in \mathbb{Z}[i]$ such that $||d|| > 0$
outputs
$q, r \in \mathbb{Z}[i]$ such that $z = qd + r$ and $||r|| < ||d||$
do
let $r = z$, $q = 0$
while $z - qr$ grows smaller
add/subtract 1 to/from q, as appropriate
while $z - qr$ grows smaller
add/subtract i to/from q, as appropriate
return q, r

Sage code

```
def divide_gaussian_integers(z, d):
    r, q = z, 0
```

Sage code

```
def divide_gaussian_integers(z, d):
    r, q = z, 0
    # which real way to step?
    if norm(r - d) < norm(r):
        s = 1
    else:
        s = -1
```

Sage code

```
def divide_gaussian_integers(z, d):
    r, q = z, 0
    # which real way to step?
    if norm(r - d) < norm(r):
        s = 1
    else:
        s = -1
    # loop to step
    while norm(r - s*d) < norm(r):
        q = q + s
        r = r - s*d
```

Sage code

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def divide_gaussian_integers(z, d):
    r, q = z, 0
    # which real way to step?
    if norm(r - d) < norm(r):
        s = 1
    else:
        s = -1
    # loop to step
    while norm(r - s*d) < norm(r):
        q = q + s
        r = r - s*d
    # which imaginary way to step?
    if norm(r - I*d) < norm(r):
        s = I
    else:
        s = -I
```

Sage code

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def divide_gaussian_integers(z, d):
    r, q = z, 0
    # which real way to step?
    if norm(r - d) < norm(r):
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        q = q + s
        r = r - s*d
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    if norm(r - I*d) < norm(r):
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    else:
        s = -I
    # loop to step
    while norm(r - s*d) < norm(r):
        q = q + s
        r = r - s*d
    return q, r
```

Example

```
sage:  divide_gaussian_integers(8 + 7*I, 2 + I)
(I + 5, -1)
```

Outline

Summary

Two types of loops

- definite: n repetitions known at outset
  - **for** $c \in C$
    - collection C of n elements controls loop
    - don't modify C
- indefinite: number of repetitions not known at outset
  - **while** *condition*
    - Boolean *condition* controls loop

Awesome mathematics!

- Newton's method
- Gaussian integers
  - division, too!