MAT 305: Mathematical Computing

Introduction to Sage

John Perry

University of Southern Mississippi

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Outline

Outline

Sage?

- Software for Algebra and Geometry Exploration
- Computer Algebra System “started” by William Stein

“Computer algebra system”?

Numerical computing

goal approximate computation, “accurate estimation”

Symbolic computing

goal exact computation

“Computer algebra system”?

Numerical computing

goal	approximate computation, “accurate estimation”
tools	floating-point numbers, vectors, matrices

Symbolic computing

goal	exact computation
tools	exact numbers, sets, abstract structures

"Computer algebra system"?

Numerical computing

- **goal** approximate computation, "accurate estimation"
- **tools** floating-point numbers, vectors, matrices
- **challenge** overflow
  - division by a small number

Symbolic computing

- **goal** exact computation
- **tools** exact numbers, sets, abstract structures
- **challenge** complexity
  - adding many fractions

Pros & cons: symbolic

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}=\frac{247}{210}$$

- summands: two digits each, *but*
- sum: 6 digits
- imagine this done thousands or millions of times

"Expression swell"

Pros & cons: symbolic

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}=\frac{247}{210}$$

- summands: two digits each, *but*
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"Expression swell"

```
sage: 1 + 10^(-5) - 1
1/100000
```

...not bad!

Pros & cons: numeric

$$0.5000 + 0.3333 + 0.2000 + 0.1429 \approx 1.176$$

- start and end with four digits, *but*
- small loss in precision

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```
sage: 1.0 + 10.0^(-5.0) - 1.0
0.0000100000000000655
sage: 1.0 + 10.0^(-15.0) - 1.0
1.11022302462516e-15
sage: 1.0 + 10.0^(-20.0) - 1.0
0.000000000000000
```

More cons: numeric

$$\begin{pmatrix} \frac{1001}{2001} & -\frac{1000}{2001} \\ -\frac{1000}{2001} & \frac{1001}{2001} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

but

$$\begin{pmatrix} \frac{1001}{2001} & -\frac{1000}{2001} \\ -\frac{1000}{2001} & \frac{1001}{2001} \end{pmatrix} \begin{pmatrix} 1.1 \\ -0.9 \end{pmatrix} = \begin{pmatrix} 201.1 \\ 199.1 \end{pmatrix}$$

- small change in input, *but*
- large change in output
- consider the effect of roundoff error…

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- small change in input, *but*
- large change in output
- consider the effect of roundoff error…

"It makes me nervous to fly an airplane since I know they are designed using floating-point arithmetic."

— Alston Householder

Why Sage?

- Free
- Cutting edge
- Access to other CAS's
  - Calculus: Maxima, SymPy, ...
  - Linear Algebra: M4RI, Linbox, PARI, ...
  - Commutative Algebra: SINGULAR, Macaulay, ...
  - Group theory: GAP, ...
  - etc.

Why Sage?

“Free” software

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- “Free as in beer”:
    - no cost to download
    - no cost to copy
    - no cost to upgrade

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  - no cost to upgrade
- “Free as in speech”:
  - no secret algorithms
  - can study implementation
  - can correct, improve, contribute

Analogy: "Free" Mathematics

Theorem (Euclid)

There are infinitely many primes.

Proof.

- Consider finite list of primes, $q_1, q_2, \ldots, q_n$.
- Let $p = q_1 q_2 \cdots q_n + 1$.
- **Fact:** since $p \neq 1$, divisible by at least one prime
- By Division Theorem, p not divisible by any q_i (remainder 1, not 0).
- p divisible by unlisted prime q_{n+1}!
- $\therefore$ no finite list, lists all primes.

$\square$

Analogy: "Secret" mathematics

Theorem (Fermat)

If $n > 2$, the equation $a^n + b^n = c^n$ has no solution with integers $a, b, c \geq 1$.

Proof.

"I have discovered a truly marvelous proof of this, which this margin is too narrow to contain."† □

†Real quote. (to be fair: in private notes, not letter, article)

Analogy: “Proprietary” mathematics

Theorem (Mersenne)

The number

$$2^n - 1$$

is prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257.$

Proof.

Trade Secret.[†] □

[†]In fact, the “theorem” is false.

But I prefer M—!

- Fine, buy your own copy
  - good reasons exist
  - student discount available
  - I will tell you the equivalent commands
- Be warned:
  - future versions not free
  - bug fixes not free
  - after you graduate, pay full price
  - not always backwards compatible (neither is Sage, but Sage is free)

Python

- “Sage” built on/with Python
  - interface between Sage and user
- Not all *components* of Sage in Python:
  - Maxima: LISP
  - SINGULAR: C/C++
  - “kernel” “compiled” for efficiency’s sake

Advantages of Python

- Modern
    - facilities for object-oriented, functional programming
- Wide distribution, usage
    - many employers use it
      (doing well in this class makes you more attractive!)
- Flexible
    - many good packages enhance it
- Can compile for efficiency using Cython

Python $\neq$ Sage

- Some Python commands don't work in worksheet mode
  - `input()`
- Sage commands do not work in plain Python

Outline

How to get Sage

- Best: SageMathCloud
- `Download, install` to your computer
  - can tinker with/break the source code
  - Windows? need LiveCD or VirtualBox player: `www.virtualbox.org/wiki/Downloads`
  - ask nicely, & I might give you a DVD with Sage for Windows, Mac, Linux

First steps in Sage

- Log in to SageMathCloud
- Start a new project
  - name it “First Sage Assignment”
  - select type “SageMath Worksheet”
  - visit “Settings”, click “Project usage and quotas”
  - click “Network access” and “Member hosting”, then “Submit changes”
  - Now return to “First Sage Assignment” (near top)
- *If you like* (not always recommended)
  - Click “Modes”, then “Typeset output”

Working with variables

Variable: symbol that represents another value

Example

```
sage: a = 7
```

Until you change it, `a` represents 7

Symbols of symbolic computation

Indeterminate: symbol with no specific value ("unknown")

- special kind of variable
- x pre-defined
  - if value assigned, no longer indeterminate
- Need more? use `var()`
  - `var('y')` defines y
  - `var('a b c d')` defines a, b, c, d
- Use undefined variable?

```
sage: x+y+z
...
NameError: name 'z' is not defined
```

Arithmetic

operation	sage equivalent
add x, y	`x + y`
subtract y from x	`x - y`
multiply x, y	`x * y`
divide x by y	`x / y`
raise x to the yth power	`x ** y` or `x ^ y`

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- Do not omit muliplication symbol
  - `2*x` $\longrightarrow 2x$
  - `2x` $\longrightarrow$ `SyntaxError: invalid syntax`
  - possible, but dangerous, to get around this using `implicit_multiplication(True)`
- Do not neglect parentheses
  - `e**(2*x)` $\neq$ `e**2*x`
- Prefer `**` to `^` for various sordid reasons (scripting)

Example

- Sage simplifies (of course)

```
sage:  5 + 3
8
sage:  (x + 3*x**2) - (2*x - x**2)
4*x^2 - x
```

Transcendental constants, functions

number	sage symbol
e	`e`
π	`pi`

operation	sage equivalent
e^x	`e**x`
$\ln x$	`ln(x)`
$\sin x$, $\cos x$, etc.	`sin(x)`, `cos(x)`, etc.

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- `log(x)` $= \ln x \neq \log_{10} x$

Some useful operations

operation	sage equivalent
factor *expr*	`factor(`*expr*`)`
simplify *expr*	`simplify(`*expr*`)`
expand *expr*	`expand(`*expr*`)`
round *expr* to *n* decimal places	`round(`*expr*`, `*n*`)`

Examples

- Some expressions simplify automatically; many need hints

```
sage:  (x**2 - 1) / (x - 1)
(x^2 - 1)/(x - 1)
sage:  (factor(x**2 - 1)) / (x - 1)
x + 1
```

  (good reason this isn't automatic: what?)
- Expand $(x-1)\left(x^3+x^2+x+1\right)$

```
sage:  expand((x-1)*(x**3+x**2+x+1))
x^4 - 1
```

- Round e to 5 decimal places

```
sage:  round(e,5)
2.71828
```

Getting help

- Online Sage documentation (tutorial, manual, etc.) at `http://www.sagemath.org/doc/`
- These notes: `www.math.usm.edu/perry/mat305ssyy/`
  (ssyy? semester and year: `sp13`, `sp14`, `sm14`, . . .)
- Textbook: www.math.usm.edu/dont_panic
- In-Sage help: command, question mark, `<Enter>`

```
sage: round?
```

[output omitted]

- Email: `john.perry@usm.edu`

Outline

Expressions

- Use computer memory by defining *expressions* with the *assignment symbol* =

```
sage: f = x**2 - 1
```

Sage does not answer when you define an expression

- Expressions remembered until you terminate Sage

```
sage: f
x^2 - 1
```

- Can remember "structures" as well as expressions

```
sage: R = GF(7) # I'll tell you what
this is later
```

Valid names

Names for expressions ("*identifiers*") can

- contain letters (A–Z), digits (0–9), or the underscore (_) *but*
- must begin with a letter or the underscore *and*
- may not contain other character (space, tab, !@#$%^, etc.)

Using expressions

- Manipulate just like the object it represents

```
sage:  factor(f)
(x - 1)*(x + 1)
sage:  f - 3
x^2 - 4
```

- Avoid repeating computations: substitute!

```
sage:  f(x=3)
8
sage:  f(x=-1)
0
sage:  f(x=4)
15
```

Alternate method of substitution

Sometimes you should use the **dictionary** method of substitution. An example would be when an identifier stands for a variable.

```
sage: f = x**2 + y**2
sage: f(x=3)
9 + y^2
sage: f({x:3})
9 + y^2
```

This also means replace x by 3 in f

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```
sage:  f = x**2 + y**2
sage:  f(x=3)
9 + y^2
sage:  f({x:3})
9 + y^2
sage:  z = x
sage:  f(z=3)
x^2 + y^2
```

This also means replace x by 3 in f

Here we let z stand in place of x

We want to replace x by 3, but…

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sage:  f({z:3})
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```

This also means replace x by 3 in f

Here we let z stand in place of x

We want to replace x by 3, but…

This works where `f(z=3)` did not

Expressions as functions

Define function using natural notation

```
sage:  f(x) = x**2
sage:  f(2)
4
sage:  f
x |--> x^2
```

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sage:  f(2)
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sage:  f
x |--> x^2
```

Automatically defines variables!

```
sage:  f(w,z) = 4*w**2-4*z**2
sage:  f(3,2)
20
sage:  f(1,z)/z
-4*(z**2 - 1)/z
sage:  f(3,2)/z
20/z
```

Expressions as functions

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```
sage:  f(x) = x**2
sage:  f(2)
4
sage:  f
x |--> x^2
```

Functions really expressions

```
sage:  factor(f)
4*(w - z)*(w + z)
sage:  type(f)
<type 'sage.symbolic.expression.Expression'>
```

Outline

Summary

- Basic, intuitive facilities for arithmetic
- Create variables to your heart's content
- Define expressions to avoid repeating computations