MAT 305: Mathematical Computing

Cython

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Outline

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An iterative function

Let $c \in \mathbb{C}$ and $f(z) = z^2 + c$.

What happens to $|f(0)|$, $|f(f(0))|$, $|f(f(f(0)))|$, ...?

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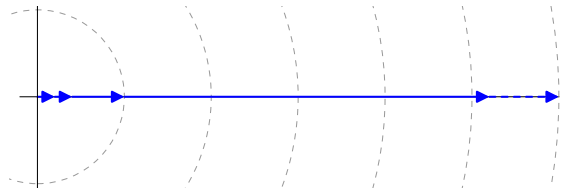
What happens to $|f(0)|, |f(f(0))|, |f(f(f(0)))|, \ldots$?

Example (can grow without bound...)

Suppose $c = 1$:

```
sage:  f(z) = z^2 + 1
sage:  a = 0
sage:  for each in xrange(10):
           a = f(a)
           print abs(a),
1 2 5 26 677 458330 210066388901
44127887745906175987802
1947270476915296449559703445493848930452791205
37918623102659260828682350280278932773702331522473885847617341507
```



An iterative function

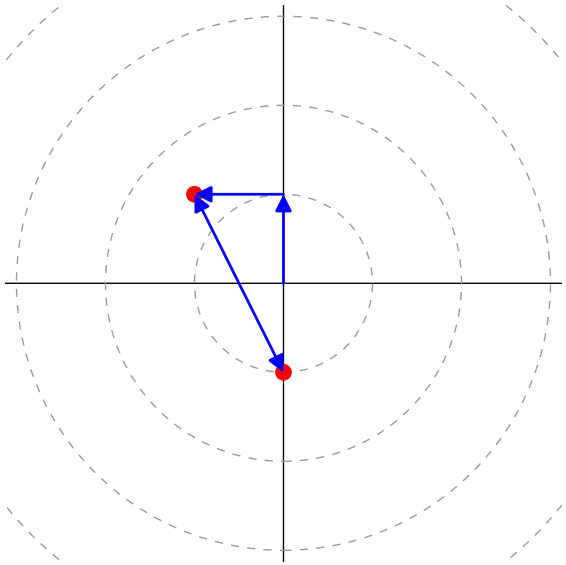
Let $c \in \mathbb{C}$ and $f(z) = z^2 + c$.

What happens to $|f(0)|, |f(f(0))|, |f(f(f(0)))|, \ldots$?

Example (. . . but not always)

Suppose $c = i$:

```
sage:  f(z) = z^2 + I
sage:  a = 0
sage:  for each in xrange(10):
           a = f(a)
           print abs(a),
1 sqrt(2) 1 sqrt(2) 1 sqrt(2) 1
sqrt(2) 1 sqrt(2)
```



Mandelbrot Numbers

Let $c \in \mathbb{C}$ and $f(z) = z^2 + c$.

Definition

The **Mandelbrot number** of c is smallest integer such that

$$\left| \underbrace{f(f(\cdots(f(0))))}_{\text{applied } n \text{ times}} \right| > 4.$$

- Notation: $\mu(c)$
- $\mu(c) = \infty$? c in **Mandelbrot set**

Examples

Example

$\mu(1) = 3$

$$f_1(0) = 0^2 + 1 = 1$$
$$f_1^2(0) = f_1(1) = 1^2 + 1 = 2$$
$$f_1^3(0) = f_1(2) = 2^2 + 1 = 5.$$

Examples

Example

$\mu(1) = 3$

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$$f_1^2(0) = f_1(1) = 1^2 + 1 = 2$$
$$f_1^3(0) = f_1(2) = 2^2 + 1 = 5.$$

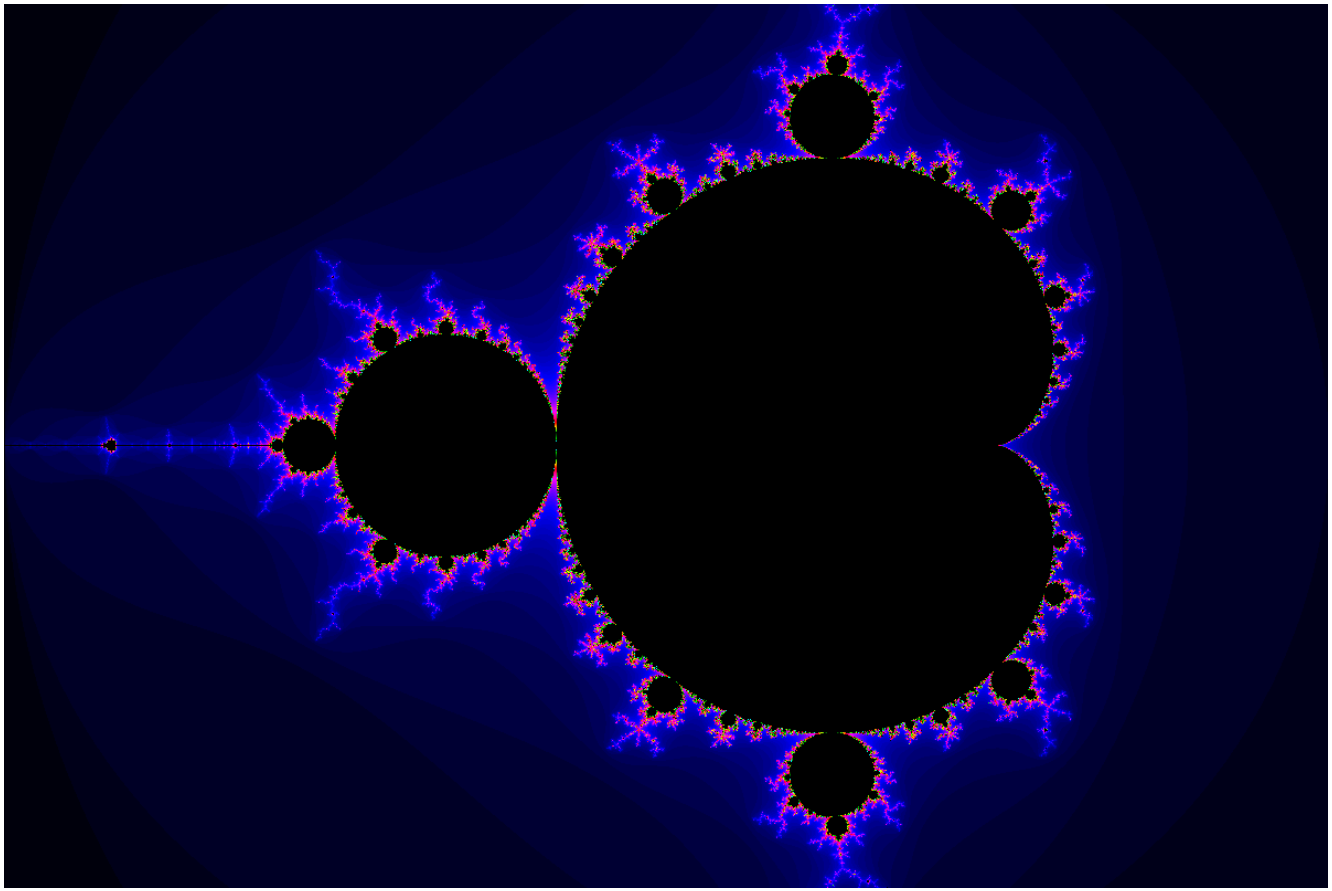
Example

$\mu(i) = \infty$

$$f_i(0) = 0^2 + i$$
$$f_i^2(0) = f_i(i) = i^2 + i = -1 + i$$
$$f_i^3(0) = f_i(i-1) = (i-1)^2 + i = -i$$
$$f_i^4(0) = f_i(-i) = (-i)^2 + i = -1 + i$$

Cool pictures

Black: Mandelbrot set. Color: iterations before too large.



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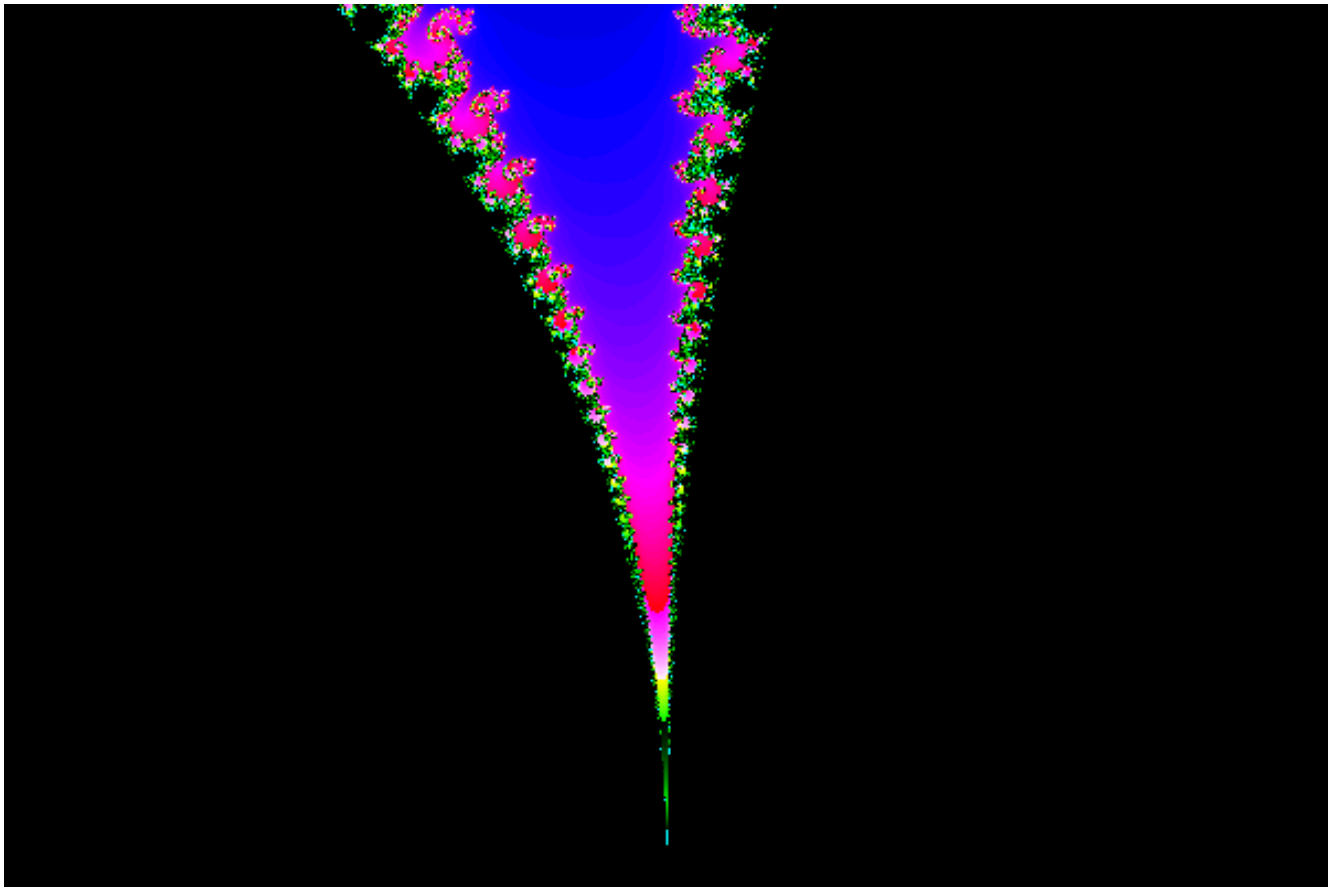

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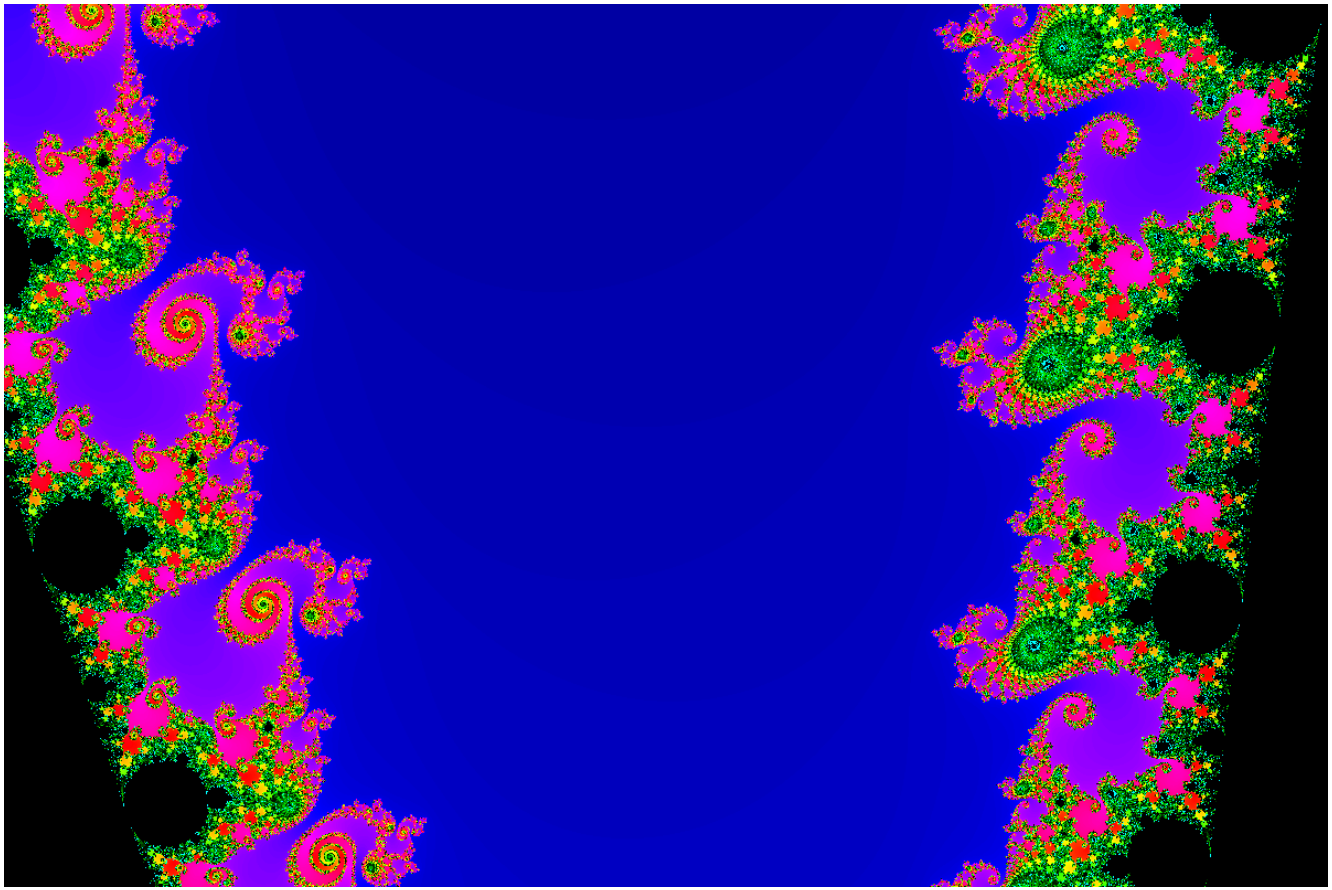

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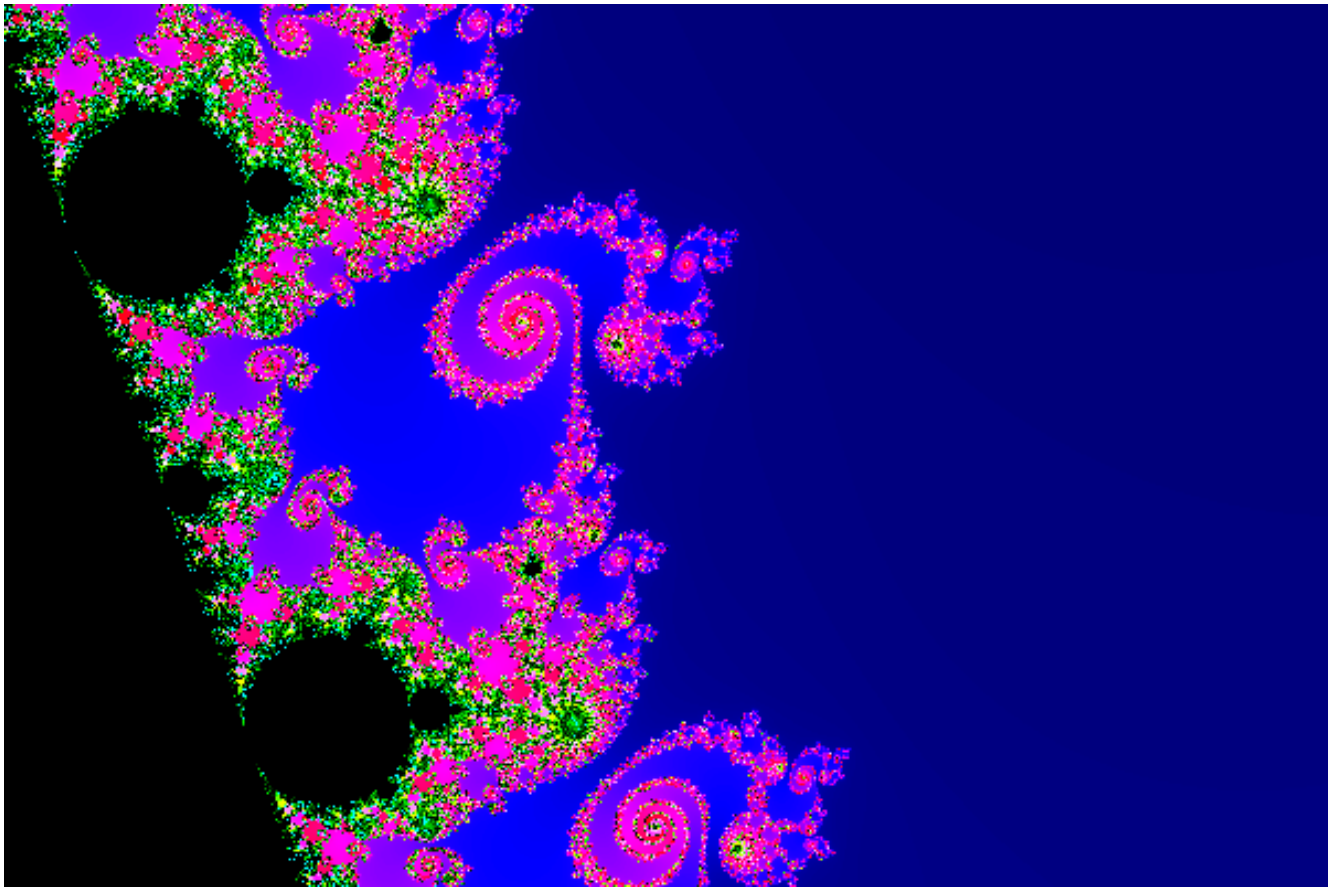

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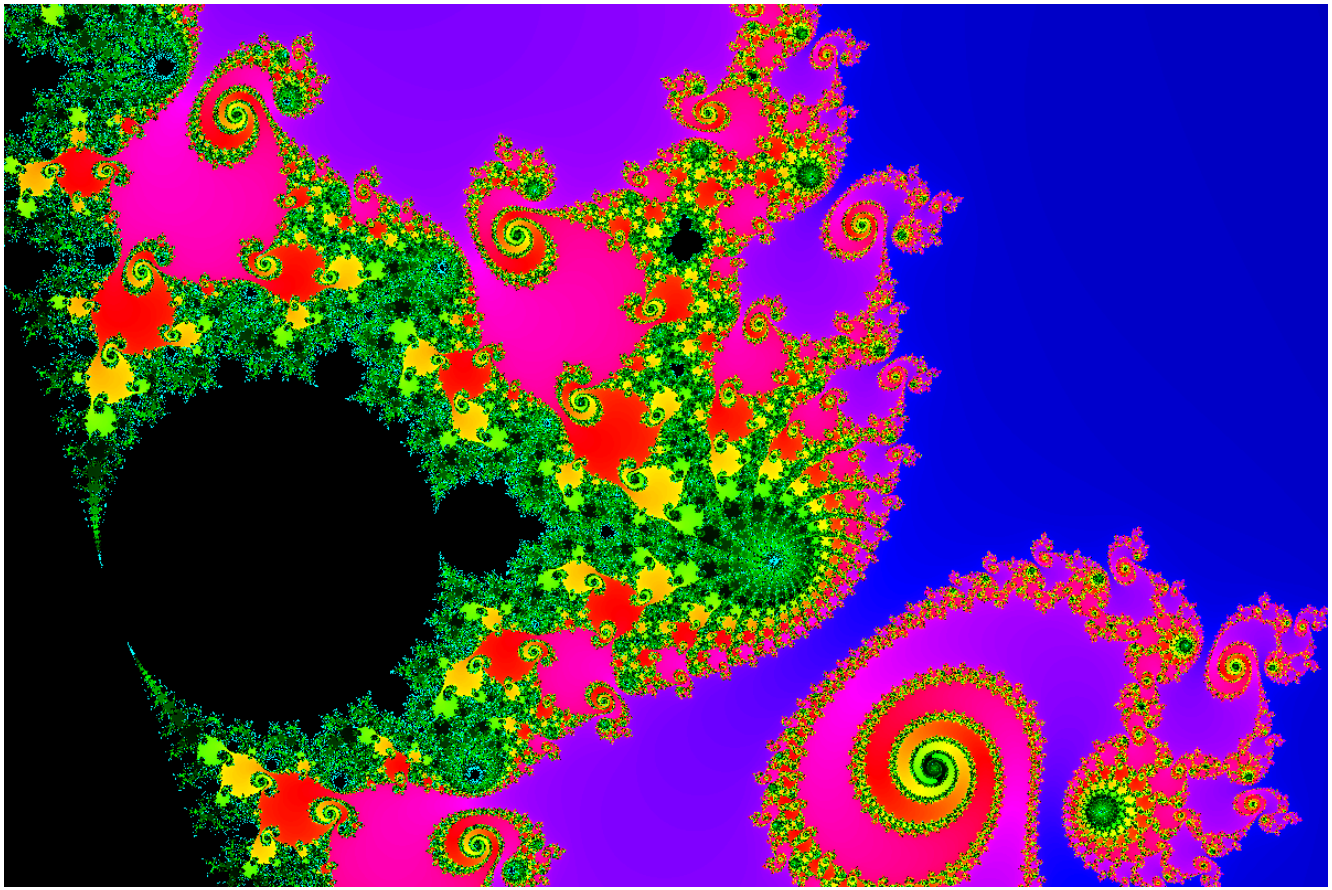

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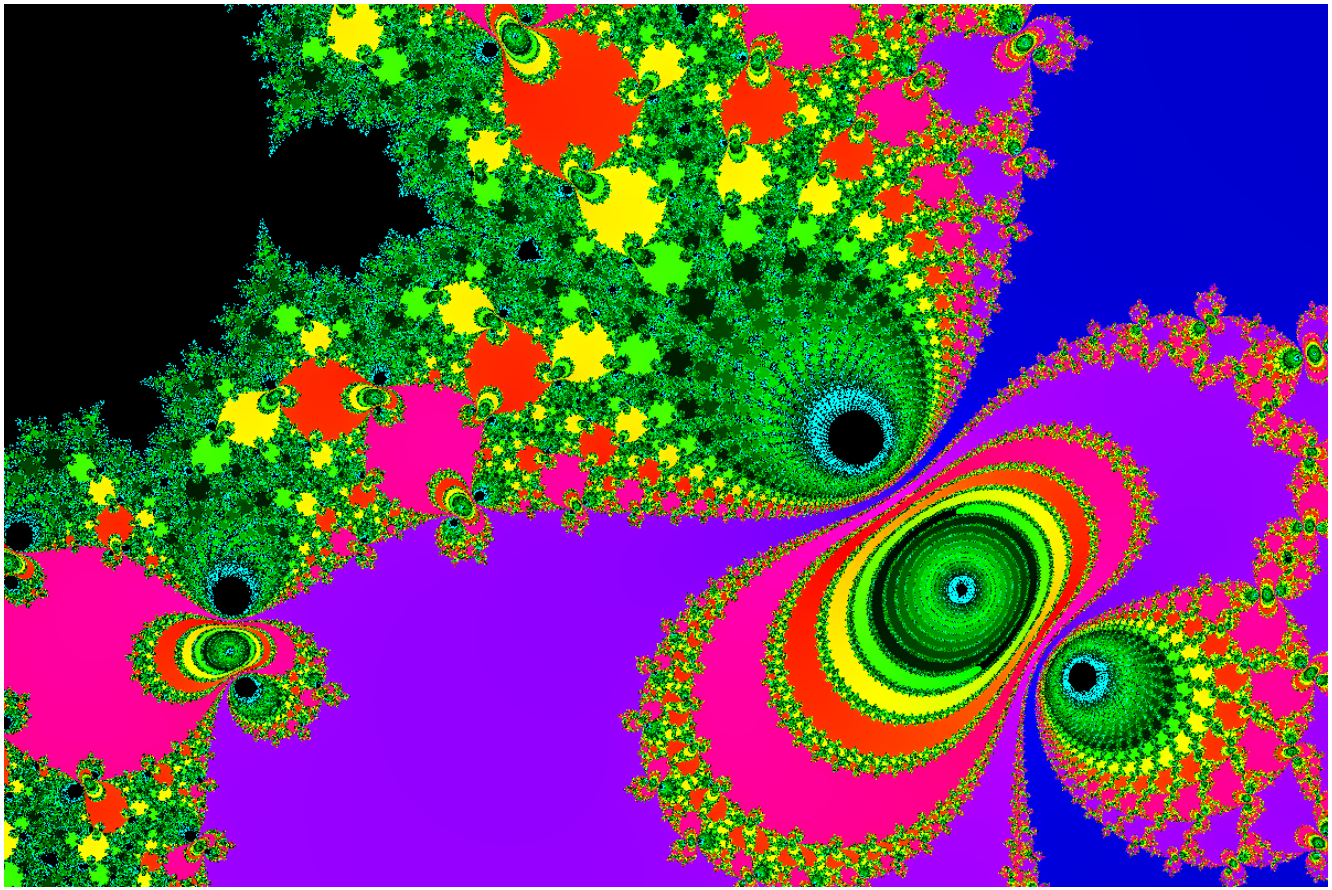

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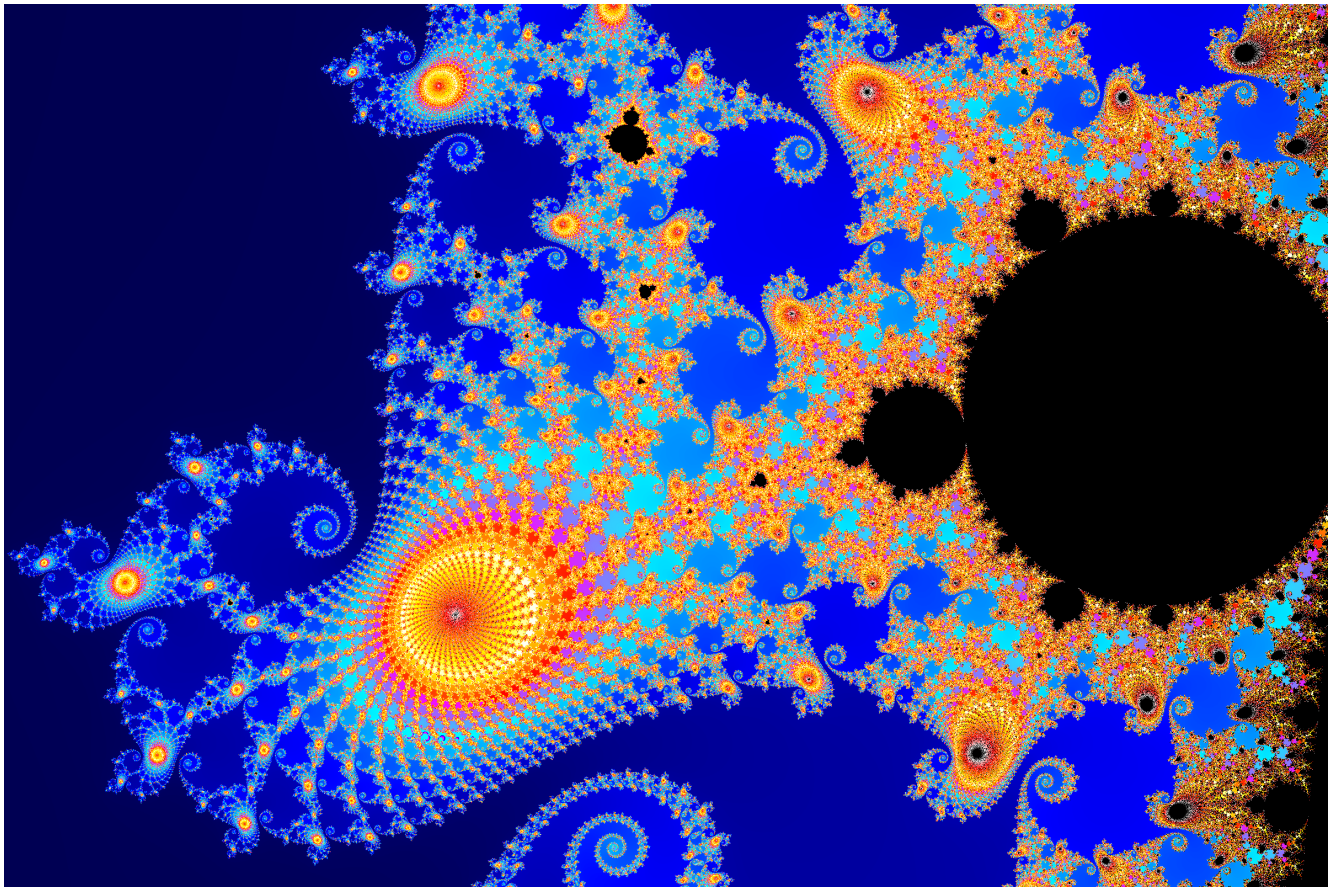

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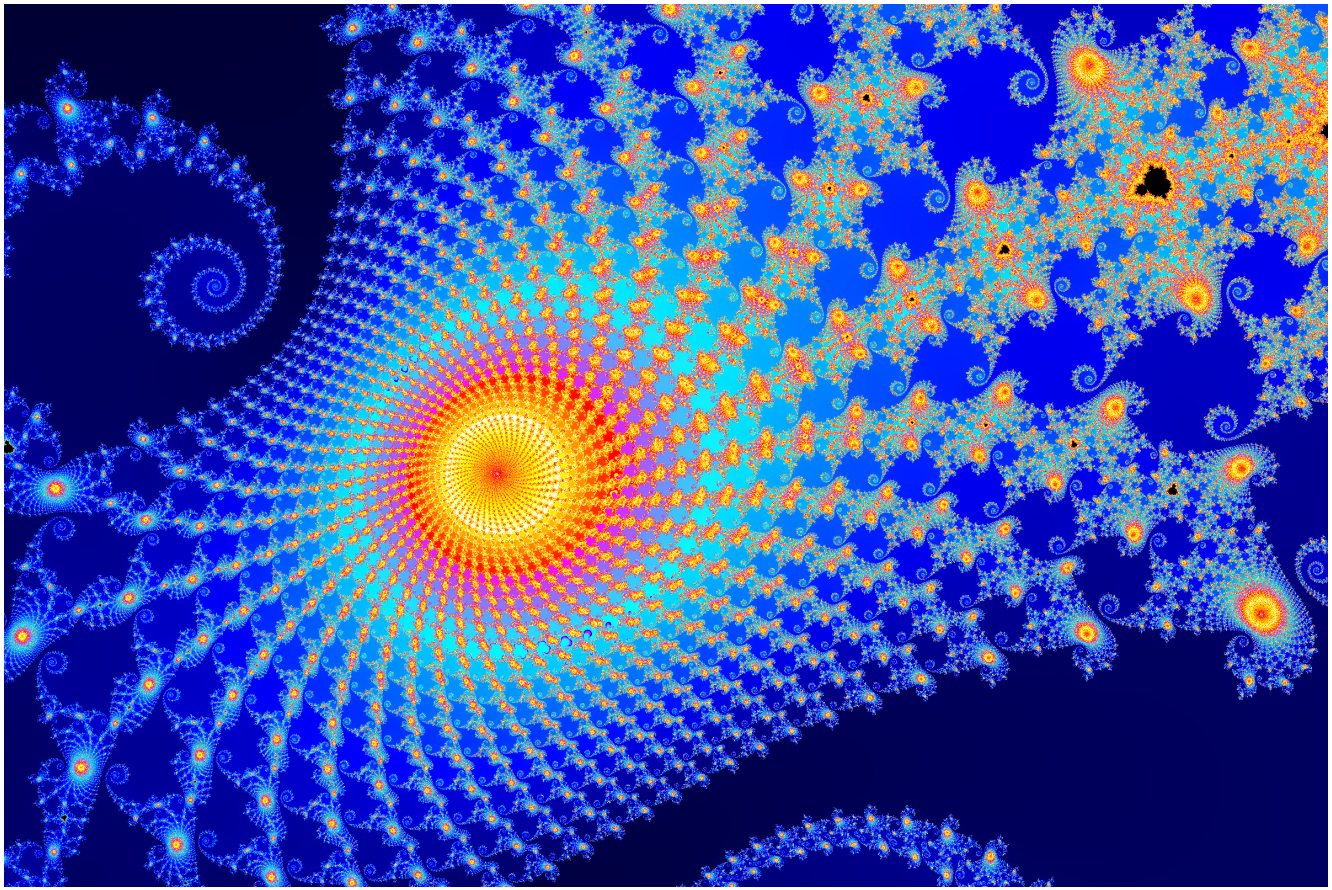

How to do this?

Challenge!

c in Mandelbrot set if

$$\lim_{n\to\infty} |f_c^n(0)| \leq 4,$$

usually impractical/impossible to check

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c in Mandelbrot set if

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usually impractical/impossible to check

How we cope

- Choose "big" $N \in \mathbb{N}^+$
- $\mu(c) \geq N$? "pretend" c in Mandelbrot set
- color c according to $\mu(c)$

Pseudocode

algorithm *Mandelbrot Number*
inputs
$c \in \mathbb{C}$
$N \in \mathbb{N}^+$
outputs
$\min(\mu(c), N)$
do
let $z = 0$
let $n = 1$
while $|z| \leq 4$ **and** $n \leq N$
let $z = z^2 + c$
increment n
return n

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$c \in \mathbb{C}$
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while $|z| \leq 4$ **and** $n \leq N$
let $z = z^2 + c$
increment n
return n

. . . of course, a lot more is needed to make a picture

First sage implementation

1. Download Python implementation of Mandelbrot
2. Attach (from **sage in terminal**, not worksheet)
   ```
   sage: %attach mandelbrot_mat305.py
   ```
3. Run
   ```
   sage: M, im = mandelbrot(optional_ymin=-1.0)
   ```
   (This will be a little slow)
4. Save image:
   ```
   sage: im.save('python_mandelbrot.png')
   ```
5. Switch to File view, click on `python_mandelbrot.png`

Problem

It's too slow.

We can fix this.

We have the technology.

Better, stronger, faster…

Second sage implementation

1. Download Cython implementation of Mandelbrot
2. Attach (command-line, not worksheet)
   ```
   sage:  %attach mandelbrot_mat305.pyx
   ```
3. Run
   ```
   sage:  M, im = mandelbrot(optional_ymin=-1.0)
   ```
   (This step should be *quite* fast now)
4. See
   ```
   sage:  im.save('cython_mandelbrot.png')
   ```
5. Switch to File view, click on `cython_mandelbrot.png`

Outline

Compiled v. Interpreted programming

Recall:

- **interpreted software**
  - computer reads one line of program
  - translates to machine code
  - *executes*, forgets translation
- **compiled software**
  - computer reads entire program
  - translates to machine code
  - *saves* translation, does not execute

Sage v. Python v. Cython

- Sage built w/Python and (some) Cython

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  - facilities for easy, efficient, elegant programming
  - many operations still slow
  - variable's type can change

Sage v. Python v. Cython

- Sage built w/Python and (some) Cython
- Python: interpreted
  - facilities for easy, efficient, elegant programming
  - many operations still slow
  - variable's type can change
- Cython: compiled
  - works with most Python
  - not standalone (runs w/Python interpreter)
  - variable's type can be unchangeable

"Type"?

Definition

kind of data variable contains

Example

integers, rounded numbers, strings, ...

"Type"?

Definition

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Example

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Python variables can contain "any data"

Example

```
sage:  a = 2
sage:  a = 'hello'
sage:  a = 3.0**5
```

"Type"?

Undeclared variables prohibited in "strongly typed" languages

compiler error in C:

```
void main() {
  a = 2;
}
```

```
test.c:2: error: 'a' undeclared (first use in this function)
```

"Type"?

Abusing type a bad idea in "strongly typed" languages

complier warning/error in C:

```
void main() {
  int a = 2;
  a = ''hello'';
}
```

```
test.c:3: warning: assignment makes integer from
pointer without a cast
```

"Type"?

Changing type prohibited in "strongly typed" languages

Compiler error in C:

```
void main() {
  int a = 2;
  char *a = ''hello'';
}
```

```
test.c:3: error: conflicting types for ‘a’
test.c:2: error: previous definition of ‘a’ was
here
```

"Type"?

Declaration has advantages and disadvantages

Disadvantages

- Can be harder to read or work
  - C++ templates border on incomprehensible
- Type often inferred or known from context
  - `x = 2.0`

Advantages

- Type known $\Longrightarrow$ no guessing
  - $2 \in \mathbb{Z}$, $2 \in \mathbb{Q}$, $2 \in \mathbb{R}$, $2 \in \mathbb{C}$, ...?
- No guessing $\Longrightarrow$ no checking $\Longrightarrow$ more efficient
  - a *lot* of behind-the-scenes work in interpreted languages

Cython's approach

declare types of functions, variables *when you want to*

- can leave some undeclared
- declare variables w/`cdef <type> <name>`
- declare functions w/`cpdef <type> <name>(...)`
  or `cdef <type> <name>(...)`
  (if you don't want to call from Python)

Types available?

- C types
  - `int`, `float`, `struct`
  - pointers: `T*`, `T**`, etc.
    - need to manage memory!
- Python types
  - `list`, `set`, `tuple`, `string`, `dict`, ...
- Sage objects
  - complicated

Compare

mandelbrot.py:

```
def compute_mandelbrot_iterates(xmin, ymin, \
    xsteps, ysteps, max_n, dx, dy):
  M = [[-1 for j in xrange(xsteps)]
  for i in xrange(ysteps)]:  ...
```

Compare

mandelbrot.pyx:

```
cdef list compute_mandelbrot_iterates(float xmin, \
    float ymin, int xsteps, int ysteps, \
    int max_n, float dx, float dy):
  cdef int i, j, n
  cdef float x, y, x0, y0, xtemp
  cdef list M = [[-1 for j in xrange(xsteps)] \
    for i in xrange(ysteps)]
  for i in xrange(ysteps):  ...
```

Do I need to write a script file?

No! Can compile Sage code in any cell by placing

```
%cython
```

before Cython function

Example of Cython in worksheet

```
sage:  %cython
       cpdef float c_square(float x):
           return x**2
       cpdef float cRiemann_left(float a, float b,
                int n):
           cdef int i
           cdef float Delta_x, xi, S
           Delta_x = (b-a)/n
           S = 0.0
           for i in xrange(n):
               xi = a + i*Delta_x
               S += c_square(xi)*Delta_x
           return S
```

Let's try it

```
sage:  %time cRiemann_left(0.0, 10.0, 100000)
333.3279113769531
CPU time:  0.00 s, Wall time:  0.00 s
```

Let's try it

```
sage:  %time cRiemann_left(0.0, 10.0, 100000)
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CPU time:  0.00 s, Wall time:  0.00 s
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Compare to a "Pythonic" version:

```
sage:  %time pRiemann_left(0.0, 10.0, 100000)
333.328333350000
CPU time:  0.23 s, Wall time:  0.23 s
```

Let's try it

```
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sage:  %time pRiemann_left(0.0, 10.0, 100000)
333.328333350000
CPU time:  0.23 s, Wall time:  0.23 s
```

The Cython version is appreciably slower.

Visualizing the Python load

You can see the C source code produced, along with an indication of Python-intensive lines

- command line: `sage -cython -a filename`
- worksheet: after entering cell, click on link labeled, `Auto-generated code...`

Example

```
 01:
+02: include "interrupt.pxi"  # ctrl-c interrupt block support
 03: include "stdsage.pxi"
 04:
 05: include "cdefs.pxi"
 06: from libc.math cimport sin
+07: cpdef float c_square(float x):
+08:     return x**2
+09: cpdef float c_Riemann_left(float a, float b, int n):
 10:     cdef int i
 11:     cdef float Delta_x, xi, S
+12:     Delta_x = (b - a)/n
+13:     S = 0.0
+14:     for i in xrange(n):
+15:         xi = a + i*Delta_x
+16:         S += c_square(xi)*Delta_x
+17:     return S
```

Example

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Why is there Python code in line 12? Click on it to expand!

Example

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 10:     cdef int i
 11:     cdef float Delta_x, xi, S
+12:     Delta_x = (b - a)/n
   __pyx_t_1 = (__pyx_v_b - __pyx_v_a);
   if (unlikely(__pyx_v_n == 0)) {
     PyErr_SetString(PyExc_ZeroDivisionError, "float division");
     {__pyx_filename = __pyx_f[0]; __pyx_lineno = 12; __pyx_cline
   }
   __pyx_v_Delta_x = (__pyx_t_1 / __pyx_v_n);
+13:     S = 0.0
+14:     for i in xrange(n):
+15:         xi = a + i*Delta_x
+16:         S += c_square(xi)*Delta_x
+17:     return S
```

Why is there Python code in line 12? To avoid division by 0!

. . . and a lot more, too!

- linking to code written in C, C++, other languages
- extending Python, Sage w/efficient data types, routines
- & more!

Outline

Summary

- Compilation can improve performance of code
- Sage uses Cython to compile code
- Cython can use data types to improve performance