MAT 305: Mathematical Computing

Linear algebra

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Outline

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Vectors

`vector(`*ring*`,` *entries*`)` where

- *ring* is base ring of *entries* (a list)
- default ring: appropriate to entries ($\mathbb{Z}$ for integers)

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Example

```
sage:  u = vector([0, 2, 2, 0])
sage:  v = vector([1, 3, -1, 2])
sage:  u + v
(1, 5, 1, 2)
sage:  u*v
4
sage:  u.norm()
2*sqrt(2)
```

Dot product!

You can plot vectors!

`v.plot()`, with optional arguments:

- `plot_type`: `'arrow'`, `'point'`, `'step'`
- `start`: tuple, list, or vector

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Example

Illustration of vector arithmetic:

```
sage:  u = vector([1,2])
sage:  v = vector([3,-1])
sage:  u.plot(color='red')
         + v.plot(color='blue',start=u)
         + (u+v).plot(color='purple')
```

You can plot vectors!

Example

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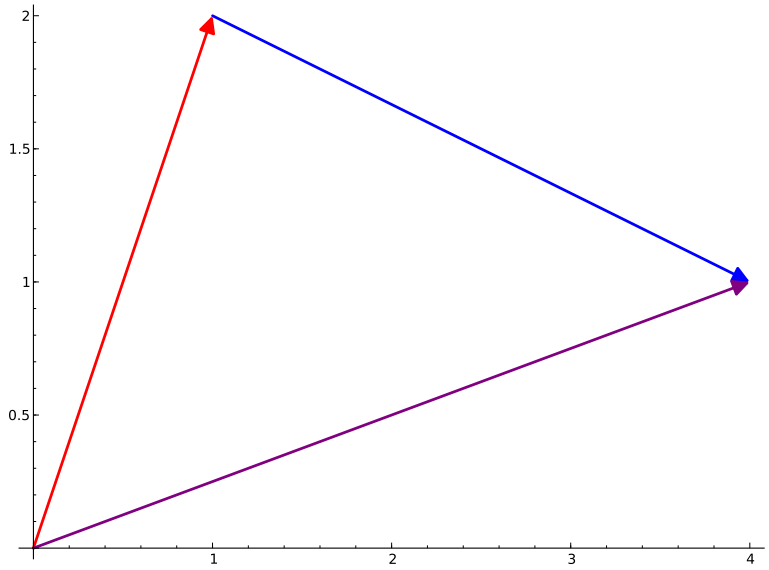
```



Outline

The `matrix()` command

`matrix(` *ring*, *#rows*, *#cols*, entries`)` where

- *ring* (optional) an appropriate algebraic ring
- *#rows*, *#cols* (optional) number of rows and columns (default depends on *entries*; no *entries* $\Longrightarrow 0 \times 0$ matrix)
- *entries* (optional) is one of
  - a list of entries, from northwest corner to southeast (if *#rows*, *#cols* specified)
  - a list of row vectors
  - none specified? all entries 0

Example matrices

```
sage:  MZ = matrix(ZZ,3,3)
sage:  MZ
[0 0 0]
[0 0 0]
[0 0 0]
```

Example matrices

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sage:  MZ = matrix(ZZ,3,3)
sage:  MZ
[0 0 0]
[0 0 0]
[0 0 0]
sage:  MR = matrix(RR,[[1,2,3],[3,2,1],[1,1,2]])
sage:  MR
[1.00000000000000 2.00000000000000 3.00000000000000]
[3.00000000000000 2.00000000000000 1.00000000000000]
[1.00000000000000 1.00000000000000 2.00000000000000]
```

Example matrices

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sage:  MZ = matrix(ZZ,3,3)
sage:  MZ
[0 0 0]
[0 0 0]
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sage:  MR = matrix(RR,[[1,2,3],[3,2,1],[1,1,2]])
sage:  MR
[1.00000000000000 2.00000000000000 3.00000000000000]
[3.00000000000000 2.00000000000000 1.00000000000000]
[1.00000000000000 1.00000000000000 2.00000000000000]
sage:  MS = matrix(SR,[[x**2 + 1, 0, 0],
       [x + I, 1, 0]])
sage:  MS
[x^2 + 1        0        0]
[  x + I        1        0]
```

Help yourself read

Good idea to put rows in different lines

```
sage:  MR = matrix(RR, [
       [1,2,3],
       [3,2,1],
       [1,1,2]
     ])
sage:  MR
[1.00000000000000 2.00000000000000 3.00000000000000]
[3.00000000000000 2.00000000000000 1.00000000000000]
[1.00000000000000 1.00000000000000 2.00000000000000]
```

Accessing matrix entries

Matrix a list of lists $\implies$ $\texttt{M[i,j]} = M_{i,j}$

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Matrix a list of lists $\implies$ `M[i,j]` $= M_{i,j}$

Example

```
sage:  MS[1,0]
x+I
sage:  MS[0,2] = x - I
sage:  MS
[x^2 + 1        0        x - I]
[  x + I        1            0]
```

(counting starts from 0)

Submatrices

- `M.submatrix(`i`,` j`,` m`,` n`)` gives
  - $m \times n$ submatrix of M
  - whose northwest corner is in row i, column j
- `M.augment(`A`)` gives $(M|A)$

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Example

```
sage:  MZ[1,1] = 1
sage:  MZ.submatrix(1,1,2,2)
[ 1 0]
[ 0 0]
```

Basic matrix operations

"dot" command	mathematics
`M.det()`	determinant
`M.inverse()`	
`M.transpose()`	
`M.eigenvalues()`	
`M.eigenvectors_right()`	right eigenvectors*
`M.eigenvectors_left()`	left eigenvectors*
`M.echelon_form()`	echelon formof unchanged M
`M.echelonize()`	change M to echelon form
`M.ncols()`	number of columns
`M.nrows()`	number of rows

*"right eigenvectors" are usual "eigenvectors"

Row arithmetic

"dot" command	mathematics
`M.set_row_to_multiple_of_row(i,j,a)`	set row i to a times row j*
`M.add_multiple_of_row(i,j,a)`	add a times row j to row i*
`M.swap_rows(i,j)`	swap rows i, j
`M.swap_columns(i,j)`	swap columns i, j

*row i changes; row j remains the same

Example: find inverse of matrix

Sage has a `.inverse()` command, but suppose you want to see steps...?

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Algorithm from High School Algebra II!

algorithm Compute inverse
  inputs
    M, an invertible matrix over a field
  outputs
    M^{-1}
  do
    Let $n = \dim(M)$
    Let A be augmented matrix $(M \mid I_n)$
    Triangularize A
    return rightmost $n \times n$ submatrix of A

Try it!

```
sage:  MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
                    [8, 3, 1, 2], [0, 1, 2, 3]])
```

Augment `MZ` *by* I_4

Try it!

```
sage:  MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
                    [8, 3, 1, 2], [0, 1, 2, 3]])
```

Augment `MZ` *by* I_4

To create I_4*, can set diagonal entries of zero matrix to 1...*

```
sage:  I4 = matrix(4,4)
sage:  for i in range(4):
           I4[i,i] = 1
sage:  I4
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

Try it!

```
sage: MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
                   [8, 3, 1, 2], [0, 1, 2, 3]])
```

Augment `MZ` *by* I_4

...or use `identity_matrix()` *command*

```
sage: I4 = identity_matrix(4)
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

Try it!

```
sage:  MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
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```

Augment MZ *by* I_4

...or use identity_matrix() *command*

```
sage:  I4 = identity_matrix(4)
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
sage:  A = MZ.augment(I4)
sage:  A
[1 2 3 4 1 0 0 0]
[0 2 2 3 0 1 0 0]
[8 3 1 2 0 0 1 0]
[0 1 2 3 0 0 0 1]
```

Try it!

```
sage: MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
                   [8, 3, 1, 2], [0, 1, 2, 3]])
```

First column: eliminate non-zero in row 3

Try it!

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                    [8, 3, 1, 2], [0, 1, 2, 3]])
```

First column: eliminate non-zero in row 3

```
sage:  A.add_multiple_of_row(2,0,-8)
sage:  A
[   1   2   3   4   1   0   0   0]
[   0   2   2   3   0   1   0   0]
[   0 -13 -23 -30  -8   0   1   0]
[   0   1   2   3   0   0   0   1]
```

Try it!

```
sage:  MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
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```

Second column: swap row w/pivot to row 2, eliminate other non-zeros

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```

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```
sage:  A.swap_rows(1,3)
sage:  A.add_multiple_of_row(0,1,-2)
sage:  A.add_multiple_of_row(2,1,13)
sage:  A.add_multiple_of_row(3,1,-2)
sage:  A
[   1   0  -1  -2   1   0   0  -2]
[   0   1   2   3   0   0   0   1]
[   0   0   3   9  -8   0   1  13]
[   0   0  -2  -3   0   1   0  -2]
```

Try it!

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sage: A
[   1   0  -1  -2   1   0   0  -2]
[   0   1   2   3   0   0   0   1]
[   0   0   3   9  -8   0   1  13]
[   0   0  -2  -3   0   1   0  -2]
```

Third column: need pivot multiply row 3 by 1/3

Try it!

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```

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```
sage:  A.set_row_to_multiple_of_row(2,2,1/3)
...
TypeError:  Multiplying row by Rational Field
element cannot be done over Integer Ring, use
change_ring or with_row_set_to_multiple_of_row
instead.
```

Try it!

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sage:  MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
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...
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```

Uh-oh! No multiplicative inverses in default ring! ($\mathbb{Z}$)
Change to $\mathbb{Q}$ *and proceed.*

Try it!

```
sage: MZ = matrix([[1, 2, 3, 4], [0, 2, 2, 3],
                   [8, 3, 1, 2], [0, 1, 2, 3]])
```

Uh-oh! No multiplicative inverses in default ring! ($\mathbb{Z}$)
Change to $\mathbb{Q}$ *and proceed.*

```
sage: A = A.change_ring(QQ)
sage: A
[   1    0   -1   -2    1    0    0   -2]
[   0    1    2    3    0    0    0    1]
[   0    0    3    9   -8    0    1   13]
[   0    0   -2   -3    0    1    0   -2]
```

Looks the same, but it's not.
Return to regularly-scheduled programming.

Try it!

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sage:  A.set_row_to_multiple_of_row(2,2,1/3)
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Third column: eliminate other non-zeros

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```
sage:  A.add_multiple_of_row(0,2,1)
sage:  A.add_multiple_of_row(1,2,-2)
sage:  A.add_multiple_of_row(3,2,2)
sage:  A
[      1      0      0      1   -5/3      0    1/3    7/3]
[      0      1      0     -3   16/3      0   -2/3 -23/3]
[      0      0      1      3   -8/3      0    1/3   13/3]
[      0      0      0      3  -16/3      1    2/3   20/3]
```

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```

Fourth column: need pivot
multiply row 4 by 1/3

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```

Fourth column: need pivot
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```
sage:  A.set_row_to_multiple_of_row(3,3,1/3)
sage:  A
[      1      0      0      1  -5/3      0   1/3   7/3]
[      0      1      0     -3  16/3      0  -2/3 -23/3]
[      0      0      1      3  -8/3      0   1/3  13/3]
[      0      0      0      1 -16/9    1/3   2/9  20/9]
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sage:  A.add_multiple_of_row(1,3,3)
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sage:  A
[      1      0      0      0    1/9   -1/3    1/9    1/9]
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[      1      0      0      0    1/9   -1/3    1/9    1/9]
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```

Have inverse! extract, test

Try it!

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```

Have inverse! extract, test

```
sage:  Minv = A.submatrix(0,4,4,4)
sage:  Minv * M
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
```

Other tools

Need another computation w/M? Remember:

- `M.<tab>` states all tools for M
- `M.<command>?` states help for `command`
- `M.<command>??` lists source code for `command`

Outline

Eigenvectors and eigenvalues

An **eigenvector** $\mathbf{x}$ of a matrix M with **eigenvalue** λ satisfies

$$M\mathbf{x} = \lambda\mathbf{x}$$

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Example

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

verification in Sage:

```
sage:  M = matrix(2,2,[0,2,2,0])
sage:  v = vector([1,-1])
sage:  M*v
(-2, 2)
```

Easy to find in Sage

```
sage:  M = matrix(2,2,[0,2,2,0])
sage:  M.eigenvectors_left()
[(2, [ (1, 1) ], 1), (-2, [ (1, -1) ], 1)]
```

What does this tell us?

- $\mathbf{e}_1 = (1,1)$ is eigenvec w/eigenval 2, mult 1
- $\mathbf{e}_2 = (1,-1)$ is eigenvec, w/eigenval -2, mult 1

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In other words,

- $M\mathbf{e}_1 = 2\mathbf{e}_1$
- $M\mathbf{e}_2 = -2\mathbf{e}_2$

Try verifying this in Sage

Neat fact of eigenvectors

Theorem (Eigendecomposition)

Let M be an $n \times n$ matrix with

- *independent eigenvectors* $\mathbf{e}_1, \ldots, \mathbf{e}_n$
- *corresponding to eigenvalues* $\lambda_1, \ldots, \lambda_n$.

We can rewrite M as $M = Q\Lambda Q^{-1}$ where

$$Q = (\mathbf{e}_1 | \mathbf{e}_2 | \cdots | \mathbf{e}_n) \qquad \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

Example

With M as defined,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 2 & \\ & -2 \end{pmatrix}$$

Verify in Sage that $M = Q\Lambda Q^{-1}$

Example

With M as defined,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 2 & \\ & -2 \end{pmatrix}$$

Verify in Sage that $M = Q\Lambda Q^{-1}$

```
sage:  Q = matrix(2,2,[1,1,1,-1])
sage:  L = matrix(2,2,[2,0,0,-2])
sage:  Q*L*Q**(-1)
[0 2]
[2 0]
```

$$\ldots \text{recall } M = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

But how is this useful?

Consider the numbers

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

But how is this useful?

Consider the numbers

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

This is the well-known Fibonacci sequence:

$$f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2}$$

Can we get a "non-recursive" formula?

Fibonacci matrix

As a matrix equation,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f_{n-1} \\ f_{n-2} \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$$

Let's try rewriting the matrix

$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

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Iterative multiplication generates the sequence

```
sage:  F = matrix(2,2,[1,1,1,0])
sage:  f12 = vector([1,1])
sage:  F*f12
[2, 1]
sage:  F^2*f12
[3, 2]
sage:  F^3*f12
[5, 3]
```

...

In short,

$$F^{n-2}\begin{pmatrix} f_2 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$$

and

$$\begin{aligned} F^{n-2} &= \left(Q\Lambda Q^{-1}\right)^{n-2} \\ &= \underbrace{\left(Q\Lambda Q^{-1}\right)\left(Q\Lambda Q^{-1}\right)\cdots\left(Q\Lambda Q^{-1}\right)}_{n-2} \\ &= \underbrace{Q\Lambda\left(Q^{-1}Q\right)\Lambda\left(Q^{-1}Q\right)\cdots\left(Q^{-1}Q\right)\Lambda Q^{-1}}_{n-2} \\ &= Q\Lambda^{n-2}Q^{-1} \end{aligned}$$

Since Λ is diagonal, it is easy to compute Λ^n

What to do?

General outline:

- Compute eigenvectors and eigenvalues

```
sage:  F.eigenvectors_right()
```

- Construct $Q\Lambda^n Q^{-1}$

```
sage:  Q = matrix(2,2,[...])
sage:  L = matrix(2,2,[...])
```

- Analyze the equation

One "drawback"

- eigenvectors, eigenvalues look inexact

```
sage:  F.eigenvectors_right()
[(-0.618033988749895?,
    [(1, -1.618033988749895?)], 1),
 (1.618033988749895?,
    [(1, 0.618033988749895?)], 1)]
```

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```

- In fact, we can determine their exact values

```
sage:  edata = F.eigenvectors_right()
sage:  lam1, lam2 = edata[0][0], edata[1][0]
sage:  lam1 = lam1.radical_expression(); lam1
-1/2*sqrt(5) + 1/2
sage:  lam2 = lam2.radical_expression(); lam2
1/2*sqrt(5) + 1/2
```

Put it together

```
[(-1/2*sqrt(5) + 1/2, [(1, -1/2*sqrt(5) - 1/2)], 1),
  (1/2*sqrt(5) + 1/2, [(1, 1/2*sqrt(5) - 1/2)], 1)]
```

```
sage:  Q = matrix(
           [1, -1/2*sqrt(5) - 1/2],
           [1,1/2*sqrt(5) - 1/2]
         )
sage:  var('n')
sage:  L = matrix(2,2,[
           (-1/2*sqrt(5) + 1/2)^(n-2), 0,
           0, (1/2*sqrt(5) + 1/2)^(n-2)
         ])
sage:  Q*L*Q**(-1)
```

...very unpleasant

...or is it?

Let $M = \left(Q\Lambda^n Q^{-1}\right)\binom{1}{1}$, and let $f_n = M_{1,1}$ (the top entry).

An "algebraic massage" (`.full_simplify()`) gives

$$f_n = \frac{\sqrt{5}}{10}\left[\left(3+\sqrt{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(3-\sqrt{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\right],$$

already a "pleasant" closed form, and thus what we wanted.

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But we can do better!

More algebraic massage...

Use Sage (in particular, `expand()`) to verify that

$$3+\sqrt{5}=2\left(\frac{1+\sqrt{5}}{2}\right)^2 \quad \text{and} \quad 3-\sqrt{5}=2\left(\frac{1-\sqrt{5}}{2}\right)^2 .$$

More algebraic massage...

Use Sage (in particular, `expand()`) to verify that

$$3+\sqrt{5}=2\left(\frac{1+\sqrt{5}}{2}\right)^2 \quad \text{and} \quad 3-\sqrt{5}=2\left(\frac{1-\sqrt{5}}{2}\right)^2 .$$

We can use this fact to rewrite

$$f_n=\frac{\sqrt{5}}{10}\left[\left(3+\sqrt{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}-\left(3-\sqrt{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\right]$$

as...

Binet's Formula

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

golden ratio

(kindly observe a moment of reverent awe)

Outline

Summary

- Sage does matrices
  - over fields *and* rings
  - symbolic ring! explore!
  - can change base ring
- You can solve some sophisticated problems using matrices on Sage