# Lab #5

MAT 305

Spring 2016

**Directions:** The usual counsels apply.

## A system of equations

1. Suppose your university ID is $abcdef$. For instance, if your ID is 123456, then $a = 1$, $b = 2$, $\ldots$, $f = 6$. Use that to define the following system of equations:

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

2. Plot each equation on the $x$-$y$ plane. In all likelihood, the two lines will not be parallel, but will intersect at exactly one point. Adjust the $x$ and $y$ axes so that this intersection is visible. *If the lines are parallel or coincident, modify one of $a$, $b$, $\ldots$, $f$ so that the lines intersect at exactly one point.*

3. Use Sage to find the *exact* solution to the system of equations. This should be a point $(x_0, y_0)$. Create a *new* plot with both lines and a big, fat point where they intersect. (Not too fat, but fat enough to see.) Make sure the point lies on top of the lines.

## An invariant

4. Define variables for $a$, $b$, $\ldots$, $f$. Use that to define the following system of equations *without any numbers at all:*

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

5. Solve the system above for $x$ and $y$.

6. You may have noticed that the solution has a common denominator. (If you didn't notice it, this would be a good time to notice.) What is sort-of-amazing, but not-really-that-amazing about that denominator?
*Hint:* Think about some basic matrix operations on the matrix that corresponds to the left sides of the original equations. It's something you should have computed in high school algebra, and *definitely* would have computed in linear algebra.

7. Suppose that $a$ and $b$ have known, concrete values. Use your answer to #6 to explain why the existence of a solution dependes *entirely* on $d$ and $e$ — and has nothing to do with $c$ and $f$!!!
   *Hint:* I'm asking about the *existence* of a solution, not the *value* of the solution once it exists. The value most certainly depends on $c$ and $f$, but the existence depends only on $d$ and $e$. So the question asks you to use the previous answer to explain this question of *existence*, not *value*.

8. Let $a$ and $b$ have the same values they had in #1. Let $g(x)$ be the denominator of the solution found in #6. Substitute these values of $a$ and $b$ into $g(x)$. You should get a linear function in two variables, $d$ and $e$. Change $d$ to $x$ and $e$ to $y$.

9. Plot the line determined (no pun intended) by this function, which has $y$-intercept `0`. Also plot the original equation $ax+by=c$. How are these two lines related? (It may not be clear unless you use the plot option `aspect_ratio=1` to make it clear.)

10. Would this relationship between the two lines hold *regardless of the values of $a$ and $b$?* That is, if the only thing you changed were $a$ and $b$ in both the line and the function $g(x)$, would the line $ax+by=c$ still have the same relationship to the corresponding value of $g(x)$?

# A point at infinity

11. We return to your original system of equations. Define lists $D$ and $E$ so that the values in $D$ move in 10 steps from $d$ *almost* to $a$ and the values in $E$ move in 10 steps from $e$ *almost* to $b$. (The difference between the final values and $a$ or $b$ should be miniscule.) For instance, if your original system is

$$\begin{cases} 1x+2y=3 \\ 4x+5y=6 \end{cases}$$

   then you could have something like $D=(4,3,2,2.5,2.1,2.01,2.001,2.0001,2.00001,2.000001)$ and $E=(5,4,3,2.5,2.1,2.01,2.001,2.0001,2.00001,2.000001)$.

12. Loop through the lists $D$ and $E$ to create a plot for each system

$$\begin{cases} ax+by=c \\ d_ix+e_iy=f \end{cases}.$$

   (Here, $d_i$ and $e_i$ refer to the $i$th elements of $D$ and $E$, respectively.) Combine the plots into a sequential animation. Animate it.

13. Describe the *eventual* relationship between the lines, especially if you let $d_i \to a$ and $e_i \to b$.

14. The field of **projective geometry** introduces a new point so that *all lines, even parallel lines, intersect at least once*. Use the animation to explain why it is appropriate to call this point a "point at infinity."
   *Hint:* You may want to adjust the minimum and maximum $x$- and $y$-values of your animation to see this more clearly.