MAT 305: Review #4

March 5, 2015

Remark. There are two problems; one on the front, one on the back. As usual, try to organize the computational cells in your worksheet under text cells that separate problems, parts of problems, add commentary, etc.

1. Define two variables a and b. Let

$$A = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix}.$$

   (a) Compute AB in Sage. Use your knowledge of trigonometry to specify a simpler form than what Sage gives. Use LaTeX to write this simpler form in a text box below the computation of AB.

   (b) Extract, and give a name to, the entry in the first row and column of AB.

$$\begin{pmatrix} \textbf{this one} & \text{not this one} \\ \text{nor this one} & \text{certainly not this one} \end{pmatrix}$$

   Try to give it a meaningful name, not just x, which is a bad idea, anyway.

   (c) In a new computational cell, type the name you just created for that entry. Then, type a dot (period). Then press tab. Look through the names that pop up for a command that might reduce the trigonometric expression to something simpler. Use that command to confirm your result in part (b). (Don't forget to add parentheses () after the command.)

   (d) Define a new matrix, C, obtained by substituting the value $a = \pi/3$ into A.

   (e) Let $\mathbf{v}$ be the vector defined by $(5,3)$.

   (f) Compute the vectors $C\mathbf{v}$, $C^2\mathbf{v}$, $C^3\mathbf{v}$, $C^4\mathbf{v}$, $C^5\mathbf{v}$.

   (g) Plot the vectors $\mathbf{v}$, $C\mathbf{v}$, $C^2\mathbf{v}$, $C^3\mathbf{v}$, $C^4\mathbf{v}$, $C^5\mathbf{v}$ in different colors (your choice). Please combine them into a single plot, rather than making six different plots.

   (h) What is the geometric effect of multiplying $\mathbf{v}$ by the matrix C repeatedly? What do you predict $C^{60}\mathbf{v}$ would look like?

2. Let

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and

$$Q = \{I_2, -I_2, \mathbf{i}, -\mathbf{i}, \mathbf{j}, -\mathbf{j}, \mathbf{k}, -\mathbf{k}\}.$$

Remark. This set is sometimes called the **quaternion group**.

(a) Show that every element of Q has an **inverse** element; that is, for every $q \in Q$, we can find $r \in Q$ such that $qr = I_2$.
Hint: Rather than retype each matrix every time, you should name the matrices Q1, Q2, etc.

(b) Show that Q is **closed** under multiplication: that is, for every $q, r \in Q$, $rq \in Q$.
Hints: Rather than retype each matrix every time, you should name the matrices Q1, Q2, etc. Rather than visually verify that a product is in Q, make Sage verify the product is in Q. A useful command appears in the lesson on Collections. Students usually lose points here because they don't check all possible products; judicious use of a loop would avoid this penalty.

Definition. We already know that matrix multiplication is associative, and you can see that the identity matrix is in Q. So, Q satisfies the **closure**, **associative**, **identity**, and **inverse** properties under its operation. Whenever a set and an operation satisfy such properties, we call that set a **group** under the operation.

Let

$$S = \{I_2, -I_2, \mathbf{j}, -\mathbf{j}\}.$$

(c) The set S actually consists of matrices of the form A above (in Problem 1). Indicate in a text box the correct value of a for each matrix. Use LaTeX.

(d) You've probably noticed that $S \subseteq Q$. Does S retain the inverse and closure properties of Q? Do you think it accurate to call S a **subgroup**?
Hint: You may not have learned what a subgroup is, but that's hardly an excuse, because the name sort of explains itself, doesn't it? After all, subgroup = subset + group...

Definition. Fix $q \in Q$. A **coset** of q with S is the set

$$qS = \{qs : s \in S\};$$

that is, qS is the set of all possible products between q and some element of S.

(e) For each $q \in Q$, compute the coset qS. (You end up with eight cosets.)

(f) Look at all the cosets you've computed. Write down anything you notice. No observation too small, but I am looking for a few in particular, so don't take this too lightly by writing just one obvious thing. Look for *all* the interesting observations you can make.