MAT 305: Final assignment

Due 5 May, 2014

Remark. As usual, try to organize the computational cells in your worksheet under text cells that separate problems, parts of problems, add commentary, etc. *Everything you need to manipulate matrices for this problem was discussed in class.*

1. The **trace** of a matrix is the sum of the elements on the main diagonal. For example, the trace of the matrix below is 4, and the main diagonal is circled.

$$\begin{pmatrix} 1 & 3 & 2 \\ -1 & -1 & -1 \\ 0 & 0 & 4 \end{pmatrix}$$

   (a) Write pseudocode to compute the trace of a matrix. Format your pseudocode properly. Your pseudocode can assume that the matrix is square.

   (b) Write a program whose input is a matrix, and whose output is the trace of the matrix. Keep in mind that:

   - Your program should be able to handle any $n \times n$ matrix, where $n = 1, 2, 3, \ldots$.
   - If the matrix is not square, then your program should indicate this somehow. At the very least, it should print a message, but it would be better to raise an exception.

2. (a) It is impossible to simplify $\int e^{-x^2} dx$ to elementary functions. Use a loop to estimate $\int_0^\infty e^{-x^2} dx$.

   (b) Previously, we wrote a program to approximate $\int_a^b f(x)\,dx$ using Left Endpoints. Adapt this to a function to approximate $\int_a^b f(x)\,dx$ using Midpoints: For the height of the i-th rectangle on the interval $[x_{i-1}, x_i]$, use the value $f(m_i)$, where $m_i = (x_{i-1} + x_i)/2$.

   (c) Write an interactive Sage application that has

   - input boxes for a function f, and endpoints $a, b \in \mathbb{R}$,
   - a slider to choose the number of approximation points N from 1 to 10,
   - a selector for either Left Endpoint or Midpoint method,

   then computes an approximation of $\int_a^b f(x)\,dx$.

3. **(Extra credit)** In a previous assignment, you wrote programs that determined whether a set S and an operation $\otimes$ satisfied the properties of closure, associativity, identity, and inverse. If there were problems with your programs, you need to fix them; see me if you need help.

   (a) Use your programs to determine which of the following satisfy all four properties.

   (i) $\mathbb{N}_5 = \{1,2,3,4\}$, where the operation is *addition* modulo 5. You learned how to define this operation in a previous assignment.

   (ii) $\mathbb{N}_5 = \{1,2,3,4\}$, where the operation is *multiplication* modulo 5.

   (iii) Choose 5 values of $n > 10$. Repeat (iii) for $\mathbb{N}_n = \{1,2,\ldots,n-1\}$. Note that the operation changes to modulo n.

   (iv) $\mathcal{P} = \{1, x, x+1\}$, where the operation is multiplication modulo 2 *and* modulo x^2+1. To set this up,

   - define a "base ring," $R = \mathbb{Z}_2$ (`sage: R = GF(2)`)
   - define a "quotient ring," $S = \mathbb{Z}_2 / \langle x^2+1 \rangle$ (`sage: S = (R[x]).quo(x^2+1)`)
   - define the set $\mathcal{P}$ (`sage: P = [S(1),S(x),S(x+1)]`)
   - define the operation $\otimes$ (`sage: def op_mul(a,b): return a*b`)

   When you test identity, use `S(`ι`)` to make sure your guess for ι has the right form.

   (v) $\mathcal{Q} = \{1, x, x+1\}$, where the operation is multiplication modulo 2 *and* modulo x^2+x+1. To set this up, repeat the above, but replace `x^2+1` by `x^2+x+1`.

   (vi) $\mathcal{F} = \{x, x^2, \sqrt{x}\}$, where the operation is composition of functions. To set this up,

   - define the set $\mathcal{F}$ (`sage: F = [x, x^2, sqrt(x)]`)
   - define the operation $\otimes$ (`sage: def op_compx(a,b): return a(x=b)`)

   To check yourself: (i) fails *only* identity; (ii) satisfies all; (iii) sometimes fails inverse; (iv) fails closure and inverse; (v) satisfies all; (vi) satisfies identity only.

   (b) Look at the results of (ii) and (iii). What characteristic was shared by all the sets that satisfied all four properties?

   (c) Try multiplying some of the elements of (iv). What very surprising result do you see? (*Note:* It is not surprising that $x^2 = 1$, because of the moduli.)