# INDIVIDUAL ASSIGNMENT 1

MAT 305 FALL 2013

## 1. Basic background

In Calculus II, you learned the definition of the **(definite) integral**,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x, \tag{1.1}$$

where $\Delta x = \frac{b-a}{n}$, and $x_i^*$ is chosen from the $i$th subinterval of $[a,b]$. You also learned the **Fundamental Theorem of Calculus**, which has two parts:

(1) If we consider the integral of $f$ as a function, then its derivative is $f$ itself. That is,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

(2) We can evaluate an integral by evaluating *any* antiderivative of $f$ at the endpoints of the interval ($a$ and $b$), and subtracting. That is, if $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This makes it important to compute the antiderivative, also called the **(indefinite) integral**.

## 2. Approximating integrals

Unfortunately, the antiderivative of an "elementary" function is not always elementary.

**Example 1.** The arclength of an ellipse cannot be computed using elementary functions, nor can

$$\int e^{x^2}\,dx.$$

In these and other cases, we need to compute the integral by approximation.

One way to approximate a definite integral is by choosing a "large" value of $n$ and *not* computing the limit in Equation (1.1) above:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x. \tag{2.1}$$

The choice of $x_i^*$ is usually one of

(1) **right-endpoint approximation:** $x_i^* = a + i\Delta x$,
(2) **left-endpoint approximation:** $x_i^* = a + (i-1)\Delta x$; and
(3) **midpoint approximation:** $x_i^* = a + \frac{2i+1}{2}\cdot\Delta x$.

Small values of $n$ give very poor approximations; "sufficiently large" values give good ones.

## 3. TECHNOLOGY TO THE RESCUE

In the lecture on Collections, you learned that Sage can produce a list fairly of numbers fairly easily, using the construction

```
[ f(xi) for xi in [x_1, x_2, ...] ]
```

where $x_1, x_2, \ldots$ are values for which you want to compute each $f(x_i)$, or

```
[ f(i) for i in range(n) ]
```

where you want to compute $f(i)$ for each $i \in \{0, 1, \ldots, n-1\}$.

In addition, Sage has a convenient command to compute the sum of a collection: `sum(C)` where $C$ is a list, set, or tuple. We can use this fact to approximate integrals "easily". Answer the following problems, and submit the results **as a Sage notebook** (to be discussed in class).

(1) Use the last digit of your student number to select the following function and interval.

| if your student number ends with... | let $f(x) = \ldots$ | and $[a,b] = \ldots$ |
|---|---|---|
| 0,1 | $\tan x^3$ | $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ |
| 2,3 | $e^{x^2}$ | $[-1, 0]$ |
| 4,5 | $\sqrt{9 + \frac{4x^2}{9 - x^2}}$ | $[0, 3]$ |
| 6,7 | $\sin(x^2)$ | $\left[0, \frac{\pi}{2}\right]$ |
| 8,9 | $\frac{\sin x}{x}$ | $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ |

(2) Create a plot that illustrates $\int_a^b f(x)\,dx$. Your plot should include
- a plot of $f(x)$ on $[a,b]$,
- a label *near one end of* $f(x)$, and
- a shading of the region between $f(x)$ and the $x$-axis.

(3) Create a plot that illustrates the approximation of $\int_a^b f(x)\,dx$ by right-endpoint estimation with four subintervals. Your plot should include
- a plot of $f(x)$ on $[a,b]$,
- a label near one end of $f(x)$,
- four filled-in rectangles that illustrate the approximation of $f(x)$, and
- four labels *inside* each rectangle, indicating the area of the rectangle.

(4) Use the `sum()` command to perform a right-endpoint approximation of the area under $f(x)$ with

(a) $n = 4$, (b) $n = 40$, (c) $n = 400$, (d) $n = 4000$, (e) $n = 40000$.

(5) Use the `sum()` command to perform a left-endpoint approximation of the area under $f(x)$ with the same values of $n$ as in the previous problem.

(6) Use the `sum()` command to perform a midpoint approximation of the area under $f(x)$ with the same values of $n$ as in the previous problem.