MAT 305: Mathematical Computing

Decision-making

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Outline

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Decision making?

A function may have to act in different ways, depending on the arguments.

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Example

Piecewise functions:

$$f(x) = \begin{cases} f_1(x), & x \in (a_0, a_1) \\ f_2(x), & x \in [a_1, a_2). \end{cases}$$

If $x \in (a_0, a_1)$, then $f(x) = f_1(x)$;
if $x \in [a_1, a_2)$, then $f(x) = f_2(x)$.

Decision making?

A function may have to act in different ways, depending on the arguments.

Example

Deciding concavity:

$$\text{If } f''(a) > 0, \text{ then } f \text{ is concave up at } x = a;$$
$$\text{if } f''(a) < 0, \text{ then } f \text{ is concave down at } x = a.$$

`if` statements

`if` *condition* `:`
  if-statement1
  if-statement2
  ...
non-if statement1

where

- *condition*: expression that evaluates to `True` or `False`
- *condition* `True`? *if-statement1*, *if-statement2*, ... performed
  - proceed eventually to *non-if statement1*
- *condition* `False`? *if-statement1*, *if-statement2*, ... skipped
  - proceed immediately to *non-if statement1*

Example

```
sage: f(x) = cos(x)
sage: ddf(x) = diff(f,2)
sage: if ddf(3*pi/4) > 0:
        print 'concave up at', 3*pi/4
concave up at 3/4*pi
```

`if-else` statements

`if` *condition*`:`

    if-statement1

    …

`else:`

    else-statement1

    …

non-if statement1

where

- *condition* `True`? *if-statement1*, … performed
    - *else-statement1*, … skipped
- condition `False`? *else-statement1*, … performed
    - *statement1*, … skipped
- proceed sooner or later to *non-if statement1*

`if-elif-else` statements

```
if condition1:
    if-statement1
    ...
elif condition2:
    elif1-statement1
    ...
elif condition3:
    elif2-statement1
    ...
...
else:
    else-statement1
    ...
non-if statement1
```

Pseudocode for `if-elif-else`

if *condition1*
    if-statement1
    …
else if *condition2*
    elseif1-statement1
    …
else if *condition3*
    elseif2-statement1
    …
…
else
    else-statement1
    …

Notice:

- indentation
- no colons
- **else if**, not **elif**

Example: concavity

Write a Sage function that tests whether a function f is concave up or down at a given point. Have it return the string `'concave up'`, `'concave down'`, or `'neither'`.

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Different choices $\Longrightarrow$ need to decide! $\Longrightarrow$ **if**

Start with pseudocode.

- inputs needed?
- output expected?
- what to do?
  - step by step
  - *Divide et impera!* **Divide and conquer!**

Pseudocode for Example

algorithm *check_concavity*

inputs

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$a \in \mathbb{R}$

$f(x)$, a twice-differentiable function at $x = a$

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`'concave up'` if f is concave up at $x = a$

`'concave down'` if f is concave down at $x = a$

`'neither'` otherwise

do

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$f(x)$, a twice-differentiable function at $x = a$

outputs

`'concave up'` if f is concave up at $x = a$

`'concave down'` if f is concave down at $x = a$

`'neither'` otherwise

do

if $f''(a) > 0$

return `'concave up'`

else if $f''(a) < 0$

return `'concave down'`

else

return `'neither'`

Try it!

```
sage: def check_concavity(a, f, x):
        ddf = diff(f, x, 2)
        if ddf(x=a) > 0:
          return 'concave up'
        elif ddf(x=a) < 0:
          return 'concave down'
        else:
          return 'neither'
```

Try it!

```
sage: def check_concavity(a, f, x):
        ddf = diff(f, x, 2)
        if ddf(x=a) > 0:
          return 'concave up'
        elif ddf(x=a) < 0:
          return 'concave down'
        else:
          return 'neither'
sage: check_concavity(3*pi/4, cos(x), x)
'concave up'
sage: check_concavity(pi/4, cos(x), x)
'concave down'
```

Example: piecewise function

Write a function whose input is any $x \in \mathbb{R}$ and whose output is

$$f(x) = \begin{cases} 1 - x^2, & x < 0 \\ 0, & x = 0 \\ x^2 - 1, & x > 0. \end{cases}$$

Example: piecewise function

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$$f(x) = \begin{cases} 1 - x^2, & x < 0 \\ 0, & x = 0 \\ x^2 - 1, & x > 0. \end{cases}$$

Three different choices $\Longrightarrow$ need to make a decision! $\Longrightarrow$ **if**

Pseudocode for example

algorithm *piecewise_f*

inputs

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$f(a)$, where f is defined as above

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Pseudocode for example

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  $a \in \mathbb{R}$

outputs

  $f(a)$, where f is defined as above

do

  if $a < 0$

    return $1 - a^2$

Pseudocode for example

algorithm *piecewise_f*

inputs

  $a \in \mathbb{R}$

outputs

  $f(a)$, where f is defined as above

do

  if $a < 0$

    return $1 - a^2$

  else if $a = 0$

    return 0

Pseudocode for example

algorithm *piecewise_f*

inputs

$a \in \mathbb{R}$

outputs

$f(a)$, where f is defined as above

do

if $a < 0$

return $1 - a^2$

else if $a = 0$

return 0

else

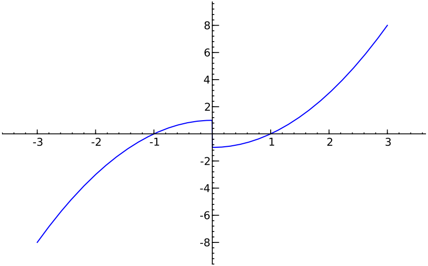
return $a^2 - 1$

Python code

```
sage: def piecewise_f(a):
        if a < 0:
          return 1 - a**2
        elif a == 0:
          return 0
        else:
          return a**2 - 1
sage: piecewise_f(3)
8
```

It gets better

```
sage:  plot(piecewise_f, xmin=-3, xmax=3)
```



It gets worse, too

How do we handle a piecewise function defined over more complicated intervals?

Example

Suppose

$$g(x)=\begin{cases}3x, & x\in[0,2)\\ -\frac{x}{3}+\frac{20}{3}, & x\in[2,20)\\ 0, & x\geq 20.\end{cases}$$

How do we define this in Sage?

Pseudocode deceptively easy

algorithm *piecewise_g*

inputs

$a \in [0, \infty)$

outputs

$g(a)$, where g is defined as above

do

if $a \in [0, 2)$

return $3a$

else if $a \in [2, 20)$

return $-\frac{a}{3} + \frac{20}{3}$

else

return 0

Pseudocode deceptively easy

algorithm *piecewise_g*

inputs

$a \in [0, \infty)$

outputs

$g(a)$, where g is defined as above

do

if $a \in [0, 2)$

return $3a$

else if $a \in [2, 20)$

return $-\frac{a}{3} + \frac{20}{3}$

else

return 0

…but how does does Sage decide $a \in [x_1, x_2)$?!?

Outline

Boolean algebra

Boolean algebra operates on only two values: {`True`, `False`}.

…or {1, 0} if you prefer

…or {Yes, No} if you prefer

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Boolean algebra operates on only two values: {`True`, `False`}.

...or {1, `0`} if you prefer

...or {Yes, No} if you prefer

Basic operations:

- **not** x
  - `True` iff x is `False`
- x **and** y
  - `True` iff both x and y are `True`
- x **or** y *("inclusive" or)*
  - `True` iff
    - x is `True`; or
    - y is `True`; or
    - both x and y are `True`

Example: and, or

```
sage: 5 > 4
True
```

obvious enough

```
sage: 5 < 4
False
```

```
sage: (5 > 4) or (5 < 4)
True
```

because at least one is True $(5 > 4)$

```
sage: (5 > 4) and (5 < 4)
False
```

because one is False

Example: not

```
sage: 4 > 4
False
```

obvious enough

```
sage: not (4 > 4)
True
sage: not ((5 > 4) or (4 < 5))
False
```

we have (not True)

```
sage: not (4 == 5)
True
```

we have (not False)

Equality and inequalities

Recall: = and == are not the same

- `x = y` assigns value of y to `x`
- `x == y` compares values of `x`, `y`, reports `True` or `False`

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- `x = y` assigns value of y to x
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For inequalities,

- `x != y` compares `x`, `y`
    - `True` iff `not (x == y)`
- `x > y`, `x < y` have usual meanings

Equality and inequalities

Recall: = and == are not the same

- `x = y` assigns value of y to x
- `x == y` compares values of x, y, reports `True` or `False`

For inequalities,

- `x != y` compares x, y
  - `True` iff `not (x == y)`
- `x > y`, `x < y` have usual meanings
- $x \geq y$? use `x >= y`
  - `True` iff `not (x < y)`
- $x \leq y$? use `x <= y`
  - `True` iff `not (x > y)`

Back to the example

Example

Suppose

$$g(x) = \begin{cases} 3x, & x \in [0,2) \\ -\frac{x}{3} + \frac{20}{3}, & x \in [2,20) \\ 0, & x \geq 20. \end{cases}$$

How do we define this in Sage? Using Boolean algebra, the pseudocode (and Python code) becomes much simpler.

Pseudocode, again

algorithm *piecewise_g*

inputs

  $a \in [0, \infty)$

outputs

  $g(a)$, where g is defined as above

do

  if $a \in [0, 2)$

    return $3a$

  else if $a \in [2, 20)$

    return $-\frac{a}{3} + \frac{20}{3}$

  else

    return 0

Pseudocode, again

algorithm *piecewise_g*

inputs

$a \in [0, \infty)$

outputs

$g(a)$, where g is defined as above

do

if $a \in [0,2)$

return $3a$

else if $a \in [2,20)$

return $-\frac{a}{3} + \frac{20}{3}$

else

return 0

…but how does does Sage decide $a \in [x_1, x_2)$?!?

use $a \geq x_1$ **and** $a < x_2$!

Sage code

```
sage: def piecewise_g(a):
        if (a >= 0) and (a < 2):
          return 3*a
        elif (a >= 2) and (a < 20):
          return -a/3 + 20/3
        else:
          return 0
```

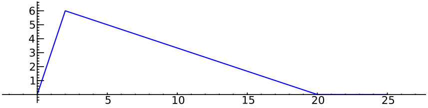
Sage code

```
sage: def piecewise_g(a):
        if (a >= 0) and (a < 2):
          return 3*a
        elif (a >= 2) and (a < 20):
          return -a/3 + 20/3
        else:
          return 0
```

Much easier to look at.

Voilà!

```
sage: def piecewise_g(a): ...
sage: pgplot = plot(piecewise_g, 0, 25)
sage: show(pgplot, aspect_ratio=1)
```



Outline

There's an error in the code

$$g(x) = \begin{cases} 3x, & x \in [0,2) \\ -\frac{x}{3} + \frac{20}{3}, & x \in [2,20) \\ 0, & x \geq 20. \end{cases}$$

What if $a < 0$?

- $g(a)$ undefined, but...
- function returns answer!

```
sage: piecewise_g(-1)
0
```

Think about

- cause?
- fix?

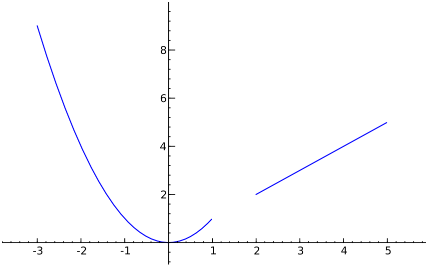
Sage has a `piecewise()` command…

`piecewise([[(`a_1`, `b_1`), `f_1`], [(`a_2`, `b_2`), `f_2`], ...])` where

- $a_i, b_i \in \mathbb{R}$
- f_i describes function on interval (a_i, b_i)

…so it's actually a little easier

```
sage:  piecewise_g = piecewise([[(-3,1), x**2],
                               [(2,5), x]])
sage:  plot(piecewise_g, xmin=-3, xmax=3)
```



Outline

Summary

- Decision making accomplished via `if-elif-else`
  - pseudocode: **if, else if, else**
- Mathematical examples abound!
  - testing properties of functions
  - piecewise functions
- Boolean algebra helps create conditions for `if` and `elif`
  - **and, or, not**
  - `<=`, `!=`, `>=`