

MAT 305: Mathematical Computing

Lecture 4: Differentiation and Integration in Sage

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Limits

Differentiation

Explicit differentiation

Implicit differentiation

Integration

Integrals

Numerical integration

Summary

① Limits

② Differentiation

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Implicit differentiation

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④ Summary

You should be in worksheet mode to repeat the examples.

Maxima, Sympy, GSL

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Summary

Maxima: symbolic Calculus

- Storied history
- Written in LISP
- Sometimes buggy in non-obvious ways

Sympy: symbolic Calculus

- Far more recent
- Written in Python
- Often slower than Maxima
- Eventual replacement for Maxima?

GSL: numerical Calculus

- GNU Scientific Library

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The `limit()` command

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Summary

`limit(f(x), x=a, direction)` where

- $f(x)$ is a function in x
- $a \in \mathbb{R}$
- $direction$ is optional, but if used has form
 - `dir='above'` or
 - `dir='below'`

Examples

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Summary

```
sage: limit(x**2-1,x=4)
```

```
15
```

```
sage: limit(x/abs(x),x=0)
```

```
und
```

(Translation: “undefined”)

```
sage: limit(x/abs(x),x=0,dir='above')
```

```
1
```

```
sage: limit(x/abs(x),x=0,dir='below')
```

```
-1
```

```
sage: limit(floor(x),x=0)
```

```
limit(floor(x),x,0)
```

(Translation: “I’m confused”)

Examples with infinite limits

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Summary

```
sage: limit(e**(-x), x=infinity)
```

0

```
sage: limit((1+1/x)**x, x=infinity)
```

e

(An indeterminate form!)

```
sage: limit((3*x**2-1)/(2*x**2+cos(x)), x=infinity)
```

3/2

```
sage: limit(ln(x)/x, x=infinity)
```

0

(Another indeterminate form!)

```
sage: limit(x/ln(x), x=infinity)
```

+Infinity

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The `diff()` command

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Summary

`diff`($f(x)$, x , n) where

- $f(x)$ is a function of x
- differentiate f with respect to x
 - “*semi-optional*”: mandatory if f has other variables
 - e.g., partial differentiation, or unknown constants
- (*optional*) compute the n th derivative of $f(x)$

Examples

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Summary

```
sage: diff(e**x)
```

```
e^x
```

```
sage: diff(x**10, 5)
```

```
30240*x^5
```

```
sage: diff(sin(x), 99)
```

```
-cos(x)
```

```
sage: var('y')
```

```
y
```

```
sage: diff(x**2+y**2-1)
```

```
...output cut...
```

```
ValueError: No differentiation variable specified.
```

```
sage: diff(x**2+y**2-1, x)
```

```
2*x
```

1 picture = 1000 words

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Summary

```
sage: f = sin(x)
sage: df = diff(f)
sage: m0 = df(x=0)
sage: p0 = point((0,f(x=0)),pointsize=30)
sage: fplot = plot(f,xmin=-pi/2,xmax=pi/2,
                  thickness=2)
sage: tan_line = plot(m0*(x-0)+f(x=0),xmin=-pi/2,
                    xmax=pi/2,rgbcolor=(1,0,0))
sage: show(p0+fplot+tan_line,aspect_ratio=1)
```

1 picture = 1000 words

Limits

Differentiation

Explicit differentiation

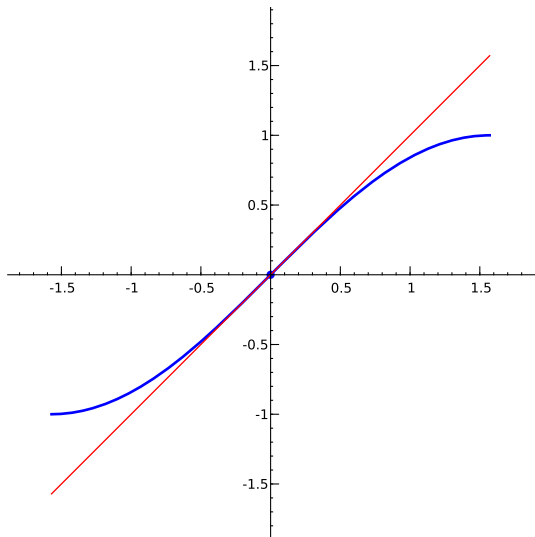
Implicit differentiation

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The `implicit_diff()` command

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Summary

There is no `implicit_diff()` command. To differentiate implicitly,

- define `yvar` as a variable using the `var()` command;
- define `yf` as an implicit function of `x` using the `function()` command;
- move everything to one side (as in implicit plots);
- differentiate the non-zero side of the equation; and
- `solve()` for `diff(yf)`

Example

```
sage: yvar = var('y')
sage: yf = function('y',x)
sage: yvar
```

y

(so yvar represents $y...$)

```
sage: yf
```

$y(x)$

(...while yf represents $y(x)$)

```
sage: diff(yf)
```

$D[0](y)(x)$

(Sage's $\frac{dy}{dx}$)

```
sage: expr = x**2 + yf**2 - 1
```

```
sage: diff(expr)
```

```
2*y(x)*D[0](y)(x) + 2*x
```

```
sage: deriv = diff(expr)
```

```
sage: solve(deriv,diff(yf))
```

```
[D[0](y)(x) == -x/y(x)]
```

...that is, $y'(x) = -\frac{x}{y}$.

Use computer memory, not yours

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Summary

```
sage: yvar = var('y')
sage: yf = function('y',x)
sage: yprime = diff(yf)
sage: deriv = diff(x**2+yf**2-1)
sage: solve(deriv,yprime)
[D[0](y)(x) == -x/y(x)]
```

1 picture = 1000 words

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```
sage: yvar = var('y')
sage: yf = function('y',x)
sage: yprime = diff(yf)
sage: f = yf**2 - x**3 + x
sage: df = diff(f)
sage: solve(df,yprime)
[D[0](y)(x) == 1/2*(3*x^2 - 1)/y(x)]
sage: mx = 1/2*(3*x^2 - 1)/yvar
sage: fplot = implicit_plot(yvar**2-x**3+x,
                           (x,-2,2),(yvar,-2,2),cmap='Reds_r')
sage: tan_line = plot(mx(x=-0.5,y=sqrt(3/8))
                     *(x+0.5)+sqrt(3/8),
                     xmin=-2,xmax=2)
sage: show(fplot+tan_line,aspect_ratio=1)
```


1 picture = 1000 words

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Explicit differentiation

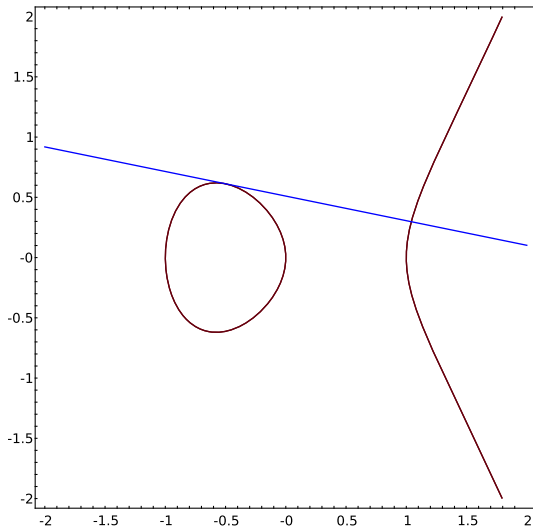
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The `integral()` command

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Summary

`integral(f(x), x, xmin, xmax)` where

- $f(x)$ is a function of the (optional) variable x
- (optional) $xmin$ and $xmax$ are limits of integration

Example

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```
sage: integral(x**2)
```

```
1/3*x^3
```

```
sage: integral(x**2,0,1)
```

```
1/3
```

```
sage: integral(1/x,1,infinity)
```

```
... output cut...
```

```
ValueError: Integral is divergent.
```

```
sage: integral(1/x**2,1,infinity)
```

```
1
```

Beware the Jabberwock...

```
sage: integral(1/x**3,1,infinity)
```

...output cut...

```
ValueError: Integral is divergent.
```

(What the—? a Maxima bug!)

(This error should not occur in Sage after version 4.1.1)

His vorpal sword in hand...

Limits

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Summary

Fortunately, Sympy works great for this integral:

```
sage: integral(1/x**3,1,infinity,  
              algorithm='sympy')
```

1/2

(Correct answer!)

Maxima bug confirmed

http://trac.sagemath.org/sage_trac/ticket/6420

- Maxima 5.13.0 was correct
 - in older versions of Sage
- Bug introduced in Maxima 5.16.3
 - Sage 4.0.2–4.1.1
- Bug fixed in Maxima 5.18.1
 - Sage 4.1.2 ← Maxima 5.19

1 picture $\stackrel{?}{=} 1000$ words

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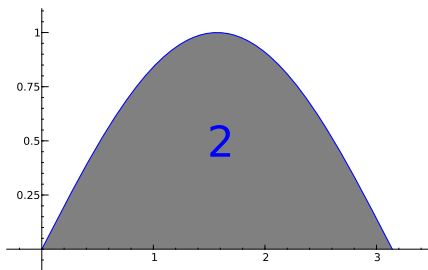
Summary

```
sage: fplot = plot(sin(x),xmin=0,xmax=pi,  
                fill='axis')
```

```
sage: farea = integral(sin(x),x,0,pi)
```

```
sage: areatext = text(farea,(pi/2,0.5),fontsize=40)
```

```
sage: fplot+areatext
```



Numerical integration: Review

Not all integrals can be simplified into elementary functions

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Numerical integration: Review

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Not all integrals can be simplified into elementary functions

Example

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int e^{-x^2} dx$$

(Gaussian error function)

```
sage: integral(e^(-x^2))  
1/2*sqrt(pi)*erf(x)
```

Numerical integration: Review

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Not all integrals can be simplified into elementary functions

Example

$$\int_{-1}^1 \sqrt{1 + \frac{4x^2}{1-x^2}} dx$$

(arclength of an ellipse)

```
sage: integral(sqrt(1+4*x**2/(1-x**2)),-1,1)
integrate(sqrt(-4*x^2/(x^2 - 1) + 1), x, -1, 1)
```

The `numerical_integral()` command

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Summary

`numerical_integral($f(x)$, $xmin$, $xmax$, $options$)` where

- $f(x)$ is a function of the defined variable x
- $xmin$ and $xmax$ are the limits of integration
- $options$ include
 - max_points , the maximum number of sample points (default: 87)

Gives two results!!!

- approximation to area
- error bound
- returned as Python tuple

Example

```
sage: numerical_integral(sqrt(1+4*x**2/(1-x**2)),  
                        -1,1)  
(4.8442240644980235, 4.5351915253605327e-06)
```

- error bound is approximately $4.535 \times 10^{-6} \approx .000004535$
- so arclength is approximately $2 \times 4.84422 = 9.68844$

Improving the estimate

```
sage: numerical_integral(sqrt(1+4*x**2/(1-x**2)),  
                           -1,1,max_points=250)  
(4.8442240644980235, 4.5351915253605327e-06)
```

- error bound is approximately $4.535 \times 10^{-6} \approx .000004535$
- so arclength is approximately $2 \times 4.84422 = 9.68844$

Doesn't seem to improve :-)

Worsening the estimate

```
sage: numerical_integral(sqrt(1+4*x**2/(1-x**2)),  
                           -1,1,max_points=10)  
(4.8363135584457568, 0.69875843576683905)
```

- error bound is approximately 0.7...!
- so arclength is somewhere on interval (4.1, 5.5)

Ouch!

Accessing only the integral

- $[i - 1]$ extracts the i th element of an ordered collection (list, tuple, etc.)
- first entry of result of `numerical_integral()` is the approximation

```
sage: app_int = numerical_integral(  
                                sqrt(1+4*x**2/(1-x**2)), -1, 1)
```

```
sage: app_int[0]  
4.8442240644980235
```


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Summary

- Sage relies on Maxima for symbolic integration and differentiation
 - Usually works fine
 - Eventual replacement: SymPy
 - Can call SymPy now
 - some things don't work
- Implicit differentiation requires some effort
 - define y as a function of x , not as a variable
- Numerical integration through GSL