MAT 305: Mathematical Computing

Lecture 11: Recursion

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Outline

You should be in worksheet mode to repeat the examples.

Outline

Recursion?

re + cursum: return, travel the path again (Latin)

Two (equivalent) views:

- **mathematical:** a function defined using itself;
- **computational:** an algorithm that invokes itself.

When recursion?

- At least one base case with no recursion
- All recursive chains terminate at base case

Proof by induction

Prove $P(n)$ for all $n \in \mathbb{N}$:

Inductive Base: Show $P(1)$

Inductive Hypothesis: Assume $P(i)$ for $1 \leq i < n$

Inductive Step: Show $P(n)$ using $P(i)$ for $1 \leq i < n$

Fibonacci's Bunnies

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- second month: pair matures;
- third month: mature pair produces new pair;
- fourth month: second pair matures, first pair produces new pair;

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- first month: one pair of bunnies;
- second month: pair matures;
- third month: mature pair produces new pair;
- fourth month: second pair matures, first pair produces new pair;
- fifth month: third pair matures, two mature pairs produce new pairs;
- …

How many pairs?

month	1	2	3	4	5	6	7	...
mature pairs								
immature pairs								
new pairs	1							
total pairs	1							

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immature pairs		1						
new pairs	1		1					
total pairs	1	1	2					

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mature pairs			1	1				
immature pairs		1		1				
new pairs	1		1	1				
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total pairs	**1**	**1**	**2**	**3**	**5**	**8**	**13**	**...**

Describing it

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- total = (# mature + # immature) + # new

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$$\therefore F_{\text{now}} = F_{\text{one month ago}} + F_{\text{two months ago}}, \text{ or}$$
$$F_i = F_{i-1} + F_{i-2}$$

$\therefore$ Fibonacci Sequence

$$F_i = \begin{cases} 1, & i = 1, 2; \\ F_{i-1} + F_{i-2}, & i \geq 3. \end{cases}$$

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Example

$$\begin{aligned} F_5 &= F_4 + F_3 \\ &= (F_3 + F_2) + (F_2 + F_1) \\ &= [(F_2 + F_1) + F_2] + (F_2 + F_1) \\ &= 3F_2 + 2F_1 \\ &= 5. \end{aligned}$$

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$$\begin{aligned} F_{100} &= F_{99} + F_{98} \\ &= \ldots \\ &= 218922995834555169026 \cdot F_2 + 135301852344706746049 \cdot F_1 \\ &= 354224848179261915075 \end{aligned}$$

Pseudocode

Easy to implement recursion:

algorithm *Fibonacci*
  inputs
    $n \in \mathbb{N}$
  outputs
    the nth Fibonacci number
  do
    if $n > 2$
      return $Fibonacci(n-2) + Fibonacci(n-1)$
    else
      return 1

Implementation

```
sage:  def fibonacci(n):
         if (n>2):
           return fibonacci(n-2) + fibonacci(n-1)
         else:
           return 1
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Implementation

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sage:  def fibonacci(n):
         if (n>2):
           return fibonacci(n-2) + fibonacci(n-1)
         else:
           return 1
sage:  fibonacci(5)
5
sage:  fibonacci(20)
6765
sage:  fibonacci(30)
832040
```

Outline

Issues in recursion

- Infinite loops
  - recursion must stop eventually

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- Infinite loops
  - recursion must stop eventually
- Wasted computation
  - `fibonacci(20)` requires `fibonacci(19)` and `fibonacci(18)`
  - `fibonacci(19)` *also* requires `fibonacci(18)`
  - $\therefore$ `fibonacci(18)` computed twice!

Example

Modify program:

```
sage:  def fibonacci(n):
         print 'computing fibonacci #', n,
         if (n>2):
           return fibonacci(n-2) + fibonacci(n-1)
         else:
           return 1
```

Example

Modify program:

```
sage:  def fibonacci(n):
         print 'computing fibonacci #', n,
         if (n>2):
           return fibonacci(n-2) + fibonacci(n-1)
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           return 1
sage:  fibonacci(5)
computing fibonacci # 5 computing fibonacci # 3
computing fibonacci # 1 computing fibonacci # 2
computing fibonacci # 4 computing fibonacci # 2
computing fibonacci # 3 computing fibonacci # 1
computing fibonacci # 2
5
```

Example

Modify program:

```
sage:  def fibonacci(n):
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sage:  fibonacci(5)
computing fibonacci # 5 computing fibonacci # 3
computing fibonacci # 1 computing fibonacci # 2
computing fibonacci # 4 computing fibonacci # 2
computing fibonacci # 3 computing fibonacci # 1
computing fibonacci # 2
5
```

$\ldots F_3$ computed 2 times; F_2, 3 times; F_1, 2 times

Workaround

Maintain list of pre-computed values:

algorithm *Fibonacci_with_table*
  globals F, a list of integers, initially `[1,1]`
  inputs
    $n \in \mathbb{N}$
  outputs
    the nth Fibonacci number
  do
    if $n > \#F$
      Let $a = Fibonacci\_with\_table(n-1)$
      Let $b = Fibonacci\_with\_table(n-2)$
      Let $F_n = a + b$
    return F_n

Implementation

```
sage:  F = [1,1]
sage:  def fibonacci_with_table(n):
         if (n>len(F)):
           print 'computing fibonacci #', n,
           a = fibonacci_with_table(n-1)
           b = fibonacci_with_table(n-2)
           F.append(a + b)
         return F[n-1]
```

Implementation

```
sage:  F = [1,1]
sage:  def fibonacci_with_table(n):
         if (n>len(F)):
           print 'computing fibonacci #', n,
           a = fibonacci_with_table(n-1)
           b = fibonacci_with_table(n-2)
           F.append(a + b)
         return F[n-1]
```

Example

```
sage:  fibonacci_with_table(5)
computing fibonacci # 5 computing fibonacci # 4
computing fibonacci # 3
5
```

However…

Avoid recursion when possible

- can often rewrite as a loop
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Example

Closed form for Fibonacci sequence:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

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Example

Closed form for Fibonacci sequence:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Coincidence? I think not...

$$\frac{1+\sqrt{5}}{2} = \textit{golden ratio}$$

Looped Fibonacci: How?

- Recursive: backwards, then forwards again
  - $F_n \longrightarrow F_{n-1}, F_{n-2} \longrightarrow \cdots \longrightarrow F_2, F_1 \longrightarrow \cdots \longrightarrow F_n$
- Looped: direct
  - $F_1, F_2 \longrightarrow F_3 \longrightarrow \cdots \longrightarrow F_n$
  - remember two previous computations
  - remember? $\Longrightarrow$ variables

Looped Fibonacci: Pseudocode

algorithm *Looped Fibonacci*

inputs

$n \in \mathbb{N}$

outputs

the nth Fibonacci number

do

— Define the base case

Let $F_{\text{prev}} = 1$, $F_{\text{curr}} = 1$

— Use the formula to move forward to F_n

Let $i = 2$

while $i < n$ **do**

— Compute next element, then move forward

Let $F_{\text{next}} = F_{\text{prev}} + F_{\text{curr}}$

Let $F_{\text{prev}} = F_{\text{curr}}$, $F_{\text{curr}} = F_{\text{next}}$

Increment i

return F_{curr}

Looped Fibonacci: Implementation

```
sage: def looped_Fibonacci(n):
        Fprev = 1
        Fcurr = 1
        i = 2
        while (i < n):
          Fnext = Fprev + Fcurr
          Fprev = Fcurr
          Fcurr = Fnext
          i = i + 1
        return Fcurr
```

Looped Fibonacci: Implementation

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         Fprev = 1
         Fcurr = 1
         i = 2
         while (i < n):
           Fnext = Fprev + Fcurr
           Fprev = Fcurr
           Fcurr = Fnext
           i = i + 1
         return Fcurr
sage:  looped_Fibonacci(100)
354224848179261915075
```

Looped Fibonacci: Implementation

```
sage:  def looped_Fibonacci(n):
         Fprev = 1
         Fcurr = 1
         i = 2
         while (i < n):
           Fnext = Fprev + Fcurr
           Fprev = Fcurr
           Fcurr = Fnext
           i = i + 1
         return Fcurr
sage:  looped_Fibonacci(100)
354224848179261915075
```

(Much faster than recursive version)

Recursive vs. Looped vs. Closed-form

- Recursive
  - pros: simpler to write
  - cons
    - slower
    - memory intensive
    - ***indefinite loop w/out loop structure***
- Looped
  - pros: not too slow, not too complicated, loop can be definite
  - cons: not (usually) as simple as recursive, sometime not obvious
- Closed-form
  - pros: one step (no loop)
  - cons: finding it often requires *significant* effort

Outline

Summary

- Recursion: function defined using other values of function
- Issues
  - can waste computation
  - can lead to infinite loops (bad design)
- Use when
  - closed/loop form too complicated
  - chains not too long
  - "memory table" feasible