TEST 3

MAT 168

Directions: Solve as many problems as well as you can in the blue examination book, writing in pencil and showing all work. Put away any cell phones; the mere appearance will give a zero.

1. (60% of test) Compute **six** of the antiderivatives indicated. (Each is worth 10%.) Some require integration by u-substitution; others, integration by parts; still others, trigonometric techniques, including trigonometric substitution; and many require multiple techniques, or the same technique applied multiple times.

   (a) $\displaystyle\int \frac{\ln^3 x}{x}\,dx$

   We'll try u-substitution with $u = \ln x$, because its derivative, $du/dx = 1/x$, also appears in the integral. We have $dx = x\,du$. Hence

$$\int \frac{\ln^3 x}{x}\,dx = \int \frac{u^3}{x}\,x\,du = \int u^3\,du = \frac{u^4}{4} + C = \frac{\ln^4 x}{4} + C\,.$$

   (b) $\displaystyle\int_1^e x^3 \ln x\,dx$

   We'll try integration by parts, because it looks like a product. The integral of $\ln x$ is hard, so we'll try

$$u = \ln x \qquad v' = x^3$$
$$u' = \frac{1}{x} \qquad v = \frac{x^4}{4}$$

   and we have

$$\begin{aligned} I = \int_1^e x^3 \ln x\,dx = \int_1^e uv'\,dx &= \left[uv - \int u'v\,dx\right]_1^e \\ &= \left[\frac{x^4 \ln x}{4} - \int \frac{1}{x}\cdot\frac{x^4}{4}\,dx\right]_1^e \\ &= \left[\frac{x^4 \ln x}{4} - \frac{1}{4}\int x^3\,dx\right]_1^e \\ &= \left[\frac{x^4 \ln x}{4} - \frac{1}{4}\cdot\frac{x^4}{4}\right]_1^e \\ &= \left(\frac{e^4 \cancelto{1}{\ln e}}{4} - \frac{e^4}{16}\right) - \left(\frac{1^4 \cancelto{0}{\ln 1}}{4} - \frac{1}{16}\right) \\ &= \frac{3e^4 - 1}{16}\,. \end{aligned}$$

Date: Spring 2018.

(c) $\displaystyle\int \sin^4 \beta \, d\beta$

The power is even, so we have to apply a half-angle formula:

$$\begin{aligned}\int \sin^4 \beta \, d\beta &= \int \left(\sin^2 \beta\right)\left(\sin^2 \beta\right) \, d\beta \\ &= \int \left(\frac{1-\cos 2\beta}{2}\right)\left(\frac{1-\cos 2\beta}{2}\right) d\beta \\ &= \frac{1}{4}\int 1 - 2\cos 2\beta + \cos^2 2\beta \, d\beta \, .\end{aligned}$$

The first two terms are relatively easy to integrate. The third term, however, has (again) an even power of cosine, so we need to use the half-angle formula again:

$$\begin{aligned}\int \sin^4 \beta \, d\beta &= \frac{1}{4}\int 1 - 2\cos 2\beta + \frac{1+\cos 4\beta}{2} \, d\beta \\ &= \frac{1}{4}\left[\beta - 2\cdot\frac{1}{2}\sin 2\beta + \frac{1}{2}\left(\beta + \frac{1}{4}\sin 4\beta\right)\right] + C \\ &= \frac{3}{8}\beta - \frac{1}{4}\sin 2\beta + \frac{1}{32}\sin 4\beta + C \, .\end{aligned}$$

(d) $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}} \, dx$

It would be unwise to try u-substitution with $u = 9 - x^2$, because its derivative, $-2x$, does *not* appear in the integral. On the other hand, we do see the form $a^2 - x^2$, which suggests that we should try the substitution $x = 3\sin\theta$. We have ${}^{dx}/_{d\theta} = 3\cos\theta$, and hence

$$\begin{aligned}\int \frac{x^2}{\sqrt{9-x^2}} \, dx &= \int \frac{9\sin^2\theta}{\sqrt{9 - 9\sin^2\theta}} \cdot 3\cos\theta \, d\theta \\ &= 9\int \frac{\sin^2\theta}{\cancel{\sqrt{9}}\sqrt{1-\sin^2\theta}} \cdot \cancel{3}\cos\theta \, d\theta \\ &= 9\int \frac{\sin^2\theta}{\cancel{\sqrt{\cos^2\theta}}} \cdot \cancel{\cos\theta} \, d\theta \\ &= 9\int \frac{1-\cos 2\theta}{2} \, d\theta \\ &= \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C \, .\end{aligned}$$

It is not enough to leave our answer in terms of θ; we must rewrite in terms of x. We use two facts. First,

$$x = 3\sin\theta \quad \Longrightarrow \quad \theta = \arcsin\frac{x}{3} \, .$$

Second,

$$\sin 2\theta = 2\sin\theta\cos\theta \, .$$

Hence

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = \frac{9}{2}\left(\theta - \frac{1}{\cancel{2}}\cdot \cancel{2}\sin\theta\cos\theta\right) = \frac{9}{2}\left(\arcsin\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right),$$

where we determine that $\cos\theta = \sqrt{9-x^2}/3$ using a right triangle whose length and hypotenuse are determine by the fact that $\sin\theta = x/3$.

(e) $\int e^{-2x}\cos 5x\,dx$

This integrand resembles a product and, as such, is an excellent candidate for integration by parts. Let

$$u = e^{-2x} \qquad v' = \cos 5x$$
$$u' = -2e^{-2x} \qquad v = \frac{1}{5}\sin 5x$$

and we have

$$\begin{aligned} I = \int e^{-2x}\cos 5x\,dx &= \int uv'\,dx \\ &= uv - \int u'v\,dx \\ &= e^{-2x}\cdot\frac{1}{5}\sin 5x - \int \left(-2e^{-2x}\right)\left(\frac{1}{5}\sin 5x\right)\,dx \\ &= \frac{1}{5}e^{-2x}\sin 5x + \frac{2}{5}\int e^{-2x}\sin 5x\,dx\,. \end{aligned}$$

We encounter another integrand that resembles a product and, as such, is an excellent candidate for integration by parts. Let

$$u = e^{-2x} \qquad v' = \sin 5x$$
$$u' = -2e^{-2x} \qquad v = -\frac{1}{5}\cos 5x$$

and we have

$$\begin{aligned} I &= \frac{1}{5}e^{-2x}\sin 5x + \frac{2}{5}\int uv'\,dx \\ &= \frac{1}{5}e^{-2x}\sin 5x + \frac{2}{5}\left(uv - \int u'v\,dx\right) \\ &= \frac{1}{5}e^{-2x}\sin 5x + \frac{2}{5}\left[\left(e^{-2x}\right)\left(-\frac{1}{5}\cos 5x\right) - \int \left(-2e^{-2x}\right)\left(-\frac{1}{5}\cos 5x\right)\,dx\right] \\ &= \frac{1}{5}e^{-2x}\sin 5x - \frac{2}{25}e^{-2x}\cos 5x - \frac{4}{25}\int e^{-2x}\cos 5x\,dx\,. \end{aligned}$$

At this point we might be tempted to panic, but you shouldn't be doing that now because you should recognize this phenomenon from all the times we encountered it before: the new integral is just I! So we can solve it as if it were a high school Algebra I problem:

$$\begin{aligned} I &= \frac{1}{5}e^{-2x}\sin 5x - \frac{2}{25}e^{-2x}\cos 5x - \frac{4}{25}I \\ \frac{29}{25}I &= \frac{1}{5}e^{-2x}\sin 5x - \frac{2}{25}e^{-2x}\cos 5x + C \\ I &= \frac{25}{29}\left(\frac{1}{5}e^{-2x}\sin 5x - \frac{2}{25}e^{-2x}\cos 5x\right) + C \\ &= \frac{5}{29}e^{-2x}\sin 5x - \frac{2}{29}e^{-2x}\cos 5x + C\,. \end{aligned}$$

(f) $\int \tan\alpha\ \sec^3\alpha\ d\alpha$

The powers are both odd, so separate out $\sec\alpha\tan\alpha$, so that we have

$$I = \int \tan\alpha\ \sec^3\alpha\ d\alpha = \int \sec^2\alpha\cdot\sec\alpha\tan\alpha\ d\alpha\,.$$

The point of this is that now we can set $u = \sec\alpha$. We have $du/d\alpha = \sec\alpha\tan\alpha$ and thus $d\alpha = du/\sec\alpha\tan\alpha$. Hence

$$I = \int u^2\cdot \cancel{\sec\alpha\tan\alpha}\,\frac{du}{\cancel{\sec\alpha\tan\alpha}} = \int u^2\,du = \frac{u^3}{3} + C = \frac{1}{3}\sec^3\alpha + C\,.$$

(g) $\int \sin^2(\sin(3\zeta))\cos(3\zeta)\ d\zeta$

This is actually a u-substitution because there's a sine inside a sine, with its derivative (cosine) on the outside. Let $u = \sin(3\zeta)$, so that $du/d\zeta = 3\cos(3\zeta)$, and we have $d\zeta = du/3\cos(3\zeta)$. Hence

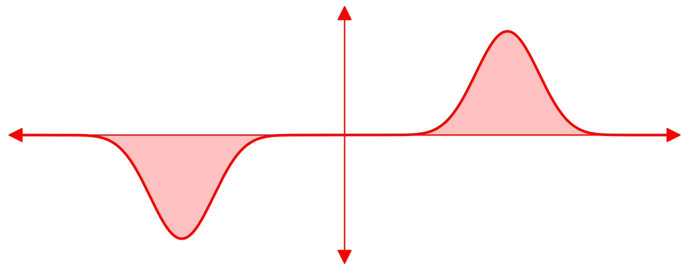
$$I = \int \sin^2(\sin(3\zeta))\cos(3\zeta)\ d\zeta = \int \sin^2 u\cdot\cancel{\cos(3\zeta)}\cdot\frac{du}{3\cancel{\cos(3\zeta)}} = \frac{1}{3}\int \sin^2 u\,du\,.$$

We apply a half-angle formula to obtain

$$\begin{aligned} I &= \frac{1}{3}\int \frac{1-\cos 2u}{2}\,du \\ &= \frac{1}{6}\int 1-\cos 2u\,du \\ &= \frac{1}{6}\left(u - \frac{1}{2}\sin 2u\right) + C \\ &= \frac{1}{6}\sin(3\zeta) - \frac{1}{12}\sin(2\sin(3\zeta)) + C\,. \end{aligned}$$

(h) $\int_{-\pi}^{\pi} \sin^{11}(\phi)\ d\phi$

Don't try to solve this problem by computing $\int \sin^{11}(\phi)\ d\phi$. It can be done, but it's an enormous waste of time. Rather, think about the geometry. At worst, you have a graphing calculator; take a glance at the graph:

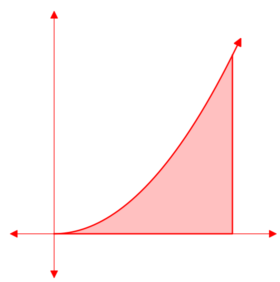


The graph is symmetric, with an area above the graph equal to the area below it. So

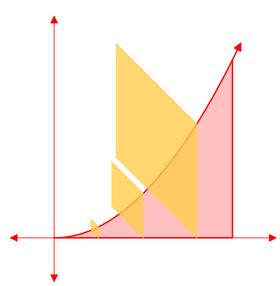
$$\int_{-\pi}^{\pi} \sin^{11}(\phi)\ d\phi = 0$$

without any tedious work at all!

2. (30% of test) Use an integral formula to set up an integral to solve each problem. **Do not evaluate the integral.**
   (a) Use the method of slicing to set up the volume of the solid whose base is the region between $y = x^2$, the x-axis, and the line $x = 2$, and whose cross-sections perpendicular to the x-axis are squares. (It will help to draw a picture, and you receive partial credit for a picture, so go ahead and try that first.)
   The region we're looking at is



We form the solid from cross-sections that look something like this:



To compute volume by slicing, compute the area of each square, which is the y-value of $f(x) = x^2$; that is,

$$V = \int_a^b A(x)\ dx = \int_0^2 y^2\, dx = \int_0^2 \left(x^2\right)^2\ dx = \int_0^2 x^4\, dx\ .$$

   (b) Use the method of disks and washers to set up the volume of the solid formed by rotating the region between $y = x^2$, the x-axis, and the line $x = 2$, about the x-axis.

The region we're looking at is the same as the one above, but this time we're rotating it about the x-axis. Since we're using disks and washers, we integrate with respect to the *same* axis as the rotation. Hence

$$V = \pi \int_a^b f(x)^2 \, dx = \pi \int_0^2 \left(x^2\right)^2 \, dx = \pi \int_0^2 x^4 \, dx \, .$$

(c) Use the method of shells to set up the volume of the solid formed by rotating the region between $y = x^2$, the x-axis, and the line $x = 2$, about the y-axis.
The region we're looking at is the same as the one above, but this time we're rotating it about the y-axis. Since we're using shells, we integrate with respect to the axis *perpendicular* to the rotation. Hence

$$V = 2\pi \int_a^b x f(x) \, dx = 2\pi \int_0^2 x \left(x^2\right) \, dx = 2\pi \int_0^2 x^3 \, dx \, .$$

3. (10% of test) Answer **one** of the two problems.
(a) For **one** of the three methods to compute the volume of a solid using integrals, explain how to derive the formula. Be sure to touch on each of the three or four common points I touched on in class and in the study guide.
I'll do volume by slicing; the others are explained similarly.
Take slices of the solid, each of width Δx. Assume the slices are a right prism; this introduces some error, but as n tends to ∞ the error will vanish. As each solid is a right prism, whose volume is $V = Ah$, where A is the area of a cross section perpendicular to the height, we have

$$V_i = A_i \cdot \Delta x$$

as the volume of the ith slice. Hence, the solid's volume is

$$V \approx \sum_{i=1}^n V_i = \sum_{i=1}^n A_i \cdot \Delta x \, .$$

To eliminate all error, let n tend to ∞ and we have the exact volume as

$$V = \lim_{n \to \infty} \sum_{i=1}^n A_i \Delta x = \int_a^b A(x) \, dx \, .$$

(b) Prove the reduction formula

$$\int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \qquad a \neq 0.$$

The integral on the left looks like a product, so we'll try integration by parts with

$$u = x^n \qquad v' = e^{ax}$$
$$u' = n x^{n-1} \qquad v = \frac{1}{a} e^{ax} \, .$$

Hence

$$\int x^n e^{ax} \, dx = \int u v' \, dx = uv - \int u' v \, dx = x^n \cdot \frac{1}{a} e^{ax} - \int \left(n x^{n-1}\right) \left(\frac{1}{a} e^{ax}\right) \, dx \, ,$$

and it is not hard to see that this simplifies immediately to the desired right-hand side.