TEST 2 FORM B

MAT 168

Directions: Solve as many problems as well as you can in the blue examination book, writing in pencil and showing all work. Put away any cell phones; the mere appearance will give a zero.

1. Compute the derivative or antiderivative, as indicated.

(a) $\frac{d}{dx}\int_0^x \tan^3 t\,dt$

Using the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_0^x \tan^3 t\,dt = \tan^3 x\,.$$

(b) $\frac{d}{dx}\int_3^{e^x} \ln^2 t\,dt$

We use the Fundamental Theorem of Calculus, but the upper limit is not simply a variable, so we have to perform a u-substitution:

$$u = e^x \quad \frac{du}{dx} = e^x \qquad .$$

Then, by the Chain Rule,

$$\frac{d}{dx}\int_3^{e^x} \ln^2 t\,dt = \ln^2(e^x)\cdot\underbrace{e^x}_{\text{Chain}} = e^x\left(\ln^2 e^x\right)\,.$$

(c) $\int x^{-1} + 3e^x\,dx$

$$\int x^{-1} + 3e^x\,dx = \ln|x| + 3e^x + C\,.$$

(d) $\int \sin 2\theta\cos^3 2\theta\,d\theta$

Let $u = \cos 2\theta$. Then ${}^{du}/_{d\theta} = -2\sin 2\theta$. Solve for $d\theta$ and we have $d\theta = -{}^{du}/_{2\sin 2\theta}$. Hence

$$\int \sin 2\theta\cos^3 2\theta\,d\theta = \int \sin 2\theta\cdot u^3\cdot\frac{du}{-2\sin 2\theta} = -\frac{1}{2}\int u^3\,du = -\frac{1}{2}\cdot\frac{u^4}{4} = -\frac{\cos^4 2\theta}{8} + C\,.$$

(e) $\int \frac{t^4 - 2\sqrt{t}}{t^2}\,dt$

Simplify, simplify.

$$\int \frac{t^4 - 2\sqrt{t}}{t^2}\,dt = \int t^2 - 2t^{-3/2}\,dt = \frac{t^3}{3} - 2\cdot\frac{t^{-1/2}}{-1/2} + C = \frac{t^3}{3} + \frac{4}{\sqrt{t}} + C\,.$$

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(f) $\displaystyle\int (\sin\theta + \csc\theta)^2 \ d\theta$

First you need to expand the product… *correctly*.

$$I := \int (\sin\theta + \csc\theta)^2 \ d\theta = \int \sin^2\theta + 2\sin\theta\csc\theta + \csc^2\theta \ d\theta \ .$$

Recall that $\csc\theta = {}^1\!/\!_{\sin\theta}$. You will also need the half-angle formula for sine.

$$I = \int \frac{1-\cos 2\theta}{2} + 2 + \csc^2\theta \ d\theta = \int \frac{5}{2} - \frac{1}{2}\cos 2\theta + \csc^2\theta \ d\theta = \int \frac{5}{2} + \csc^2\theta \ d\theta - \frac{1}{2}\int \cos 2\theta \ d\theta \ .$$

Notice that we combined the $1/2$ in the half-angle formula with the 2, then separated the integral. The reason for the separation is that it is relatively straightforward to dispatch the first "new" integral. For the second "new" integral we use the substitution $u = 2\theta$; that gives us $du/d\theta = 2$, so $d\theta = du/2$.

$$I = \frac{5}{2}\theta - \cot\theta - \frac{1}{2}\int \cos u \ \frac{du}{2} = \frac{5}{2}\theta - \cot\theta - \frac{1}{4}\sin u + C = \frac{5}{2}\theta - \cot\theta - \frac{1}{4}\sin 2\theta + C \ .$$

(g) $\displaystyle\int \frac{5}{16t^2+1} \ dt$

This has the form of an arctangent's derviative, only it's a bit different. Let $u = 4t$. Then

$$\int \frac{5}{16t^2+1} \ dt = 5\int \frac{1}{(4t)^2+1} \ dt = 5\int \frac{1}{u^2+1} \ \frac{du}{4} = \frac{5}{4}\arctan u + C = \frac{5}{4}\arctan(4t) + C \ .$$

2. For the following applied problems, use an integral formula to set up an integral to solve the problem. **Do not evaluate the integral in this problem (see below).**

(a) Set up the average value of $f(t) = \sin t$ on the interval $[\pi/4, 3\pi/4]$.

$$\overline{y} = \frac{1}{\frac{3\pi}{4} - \frac{\pi}{4}} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin t \ dt \ .$$

(b) If the marginal cost of producing n houses is $MC(n) = (n-5)^2 - 40$ thousand dollars, set up the net change in cost when production changes from 4 to 5 houses.

$$C(5) - C(4) = \int_4^5 MC(n) \ dn = \int_4^5 (n-5)^2 - 40 \ dn \ .$$

(c) Set up the total area of the region between the curves $y = 9 - x^2$, $y = x - 1$, and both axes.

This problem is a disaster: I must have introduced some typo during revision of the test. What it should really ask is for the region between the curves, **full stop**. So that's what we'll do.

But it gets worse! I had worked out an elegant formula that would guarantee the endpoints were pretty numbers, and the endpoints still end up ugly, as you will see in a moment. Anyway, don't panic; we're just setting this up.

As mentioned, we need to find where the curves intersect. So, set them equal to each other and solve for x.

$$9 - x^2 = x - 1$$
$$0 = x^2 + x - 10 \ .$$

We have a quadratic equation, so apply the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{41}}{2} \ .$$

Egad. Well, I'll write a for the left endpoint and b for the right endpoint. We still have to determine which curve is on top: substitute $x = 0$ into them and we find that the parabola is on top ($9 - 0^2 > 0 - 1$). So the total area is

$$\int_a^b \left(9 - x^2\right) - (x - 1)\ dx \quad .$$

(d) Use the method of slicing to set up the volume of the solid whose base is the region between $y = x^2 - 1$ and the x-axis and whose cross-sections perpendicular to the y-axis are squares.
This region lies *beneath* the x-axis. Since the cross-sections are perpendicular to the y-axis, we have to integrate with respect to y. The minimum y-value is -1 (the parabola's vertex), and the maximum y-value is 0 (the x-axis). (It will help to look at a picture.) The cross-sections are squares, so we're looking at

$$V = \int_{-1}^{0} s^2\ dy \quad .$$

The length of the cross-section's side at any y value is $s = 2x$, but if we integrate with respect to y we need to rewrite x in terms of y, so we think of the side length as $s = 2\sqrt{y+1}$. Hence

$$V = \int_{-1}^{0} \left(2\sqrt{y+1}\right)^2\ dy \quad .$$

(e) Use the method of disks and washers to set up the volume of the solid formed by rotating the region defined by $y = x^2 - 1$ and the x-axis about the x-axis.
The curve intersects the x-axis at $x = \pm 1$. The height of a rectangle rotated about the axis is y, so

$$V = \pi \int_{-1}^{1} y^2\ dx = \pi \int_{-1}^{1} \left(x^2 - 1\right)^2\ dx \ .$$

(f) Use the method of shells to set up the volume of the solid formed by rotating the region defined by $y = 3x - 9$ and both axes about the y-axis.
The line intersects the y-axis at $(0, -9)$ and the x-axis at $(3, 0)$. If we rotate this about the y-axis, the height of a shell will be $3x - 9$, so

$$V = 2\pi \int_0^3 x\,(3x - 9)\ dx \quad .$$

3. Choose one of the integrals you found in problem 2. Simplify it to elementary form.
   Omitted from solutions.
4. Choose one of the scenarios described in problem 2. Explain how to derive the formula you used. Be sure to touch on each of the three or four common points I touched on in class.
   See study guide for this test, which gives an example.
5. Choose *one of* problems 2(a), 2(d), or 2(f).
   2(a) Diagram f and its average value in appropriate fashion.
   Omitted from solutions.
   2(d) Sketch the solid and a cross-section.
   Omitted from solutions.
   2(f) Sketch the solid and one of its shells.
   Omitted from solutions.
6. **(Extra Credit)** If $f'(x) = g'(x)$, is $f(x) = g(x)$? Why or why not?
   No. Two functions have the same derivative if they differ by a constant, and the constant could be nonzero. For instance, $f(x) = 1$ and $g(x) = 2$ satisfy the hypothesis ($f'(x) = g'(x)$), but not the conclusion ($f(x) = g(x)$).