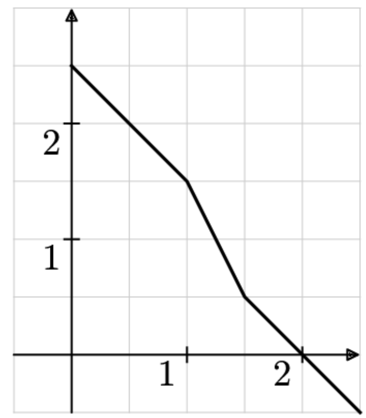
TEST 2 FORM A

MAT 168

Directions: Solve as many problems as well as you can in the blue examination book, writing in pencil and showing all work. Put away any cell phones; the mere appearance will give a zero.

1. Let $A(x) = \int_0^x f(t)\ dt$, where the graph of f is shown at right.



(a) Estimate $A(0)$, $A(1)$, and $A(2)$.

Remember that A is the *integral,* hence the *area* under the curve, from 0 to x. Hence $A(0)$ is the area from 0 to 0, which is nothing, so

$$A(0) = 0\ .$$

Meanwhile, $A(1)$ corresponds to the area of a trapezoid of "height" 1 and of "bases" 2.5 and 1.5, so

$$A(1) = \frac{1}{2} \cdot 1 \cdot (2.5 + 1.5) = 2\ .$$

(You can also count the number of boxes and divide by 4 — notice that there are 4 boxes in 1 square unit!) Finally, $A(2)$ adds to our trapezoid another trapezoid and a triangle, so

$$A(2) = A(1) + \left[\frac{1}{2} \cdot \frac{1}{2} \cdot (1.5 + .5)\right] + \left(\frac{1}{2} \cdot \frac{1}{4}\right) = 2 + .5 + .75 = 3.25\ .$$

(b) Over what interval is A increasing?

We see from the numbers that A is increasing over $[0, 2]$.

Another way of looking at this is that the graph of f always lies above the x-axis on the interval $[0, 2]$. It starts to decrease after $x = 2$ because the graph goes below the x-axis.

(c) Where does A reach its maximum?

As noted, A is increasing from $[0, 2]$, so A reaches its maximum at $x = 2$.

2. Simplify the derivative or antiderivative, as indicated.

(a) $\displaystyle\int \frac{x^5 - x}{x^2}\ dx$

This integral becomes easy if you split it into a sum fractions, which you can do when you're splitting a sum in the numerator:

$$\int \frac{x^5 - x}{x^2}\ dx = \int x^3 - \frac{1}{x}\ dx = \frac{x^4}{4} - \ln|x| + C\ .$$

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(b) $\int e^{5x}\,dx$

We can integrate e^u, so let $u = 5x$. We have

$$\frac{du}{dx} = 5 \quad \Longrightarrow \quad dx = \frac{du}{5}\,.$$

Hence

$$\int e^{5x}\,dx \quad = \quad \int e^u\,\frac{du}{5} \quad = \quad \frac{1}{5}e^u + C \quad = \frac{1}{5}e^{5x} + C\,.$$

(c) $\int_1^3 \cos\left(\frac{\pi}{2}\cdot x\right)\,dx$

We can integrate $\cos u$, so let $u = {}^{\pi x}/_2$. We have

$$\frac{du}{dx} = \frac{\pi}{2} \quad \Longrightarrow \quad dx = \frac{2\,du}{\pi}\,.$$

Moreover, the limits of integration change from

$$x = 1 \text{ to } u = \frac{\pi}{2}\cdot 1 = \frac{\pi}{2} \quad \text{and} \quad x = 3 \text{ to } u = \frac{\pi}{2}\cdot 3 = \frac{3\pi}{2}\,.$$

Hence

$$\int_1^3 \cos\left(\frac{\pi}{2}\cdot x\right) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos u\,\frac{2\,du}{\pi} = \frac{2}{\pi}\sin u\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{2}{\pi}\left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) = \frac{2}{\pi}(-1-1) = -\frac{4}{\pi}\,.$$

(d) $\int (\sin\theta + \cos\theta)^2\,d\theta$

This requires us first to expand the product:

$$I = \int (\sin\theta + \cos\theta)^2\,d\theta = \int \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\,d\theta\,.$$

Recall the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$. We use that to simplify our integral:

$$I = \int 1 + 2\sin\theta\cos\theta\,d\theta\,.$$

The integral of 1 is easy; it simply gives us θ. For the other term, let $u = \sin\theta$. We have

$$\frac{du}{d\theta} = \cos\theta \quad \Longrightarrow \quad d\theta = \frac{du}{\cos\theta}\,.$$

Hence

$$I = \theta + \int 2u\cancel{\cos\theta}\cdot\frac{du}{\cancel{\cos\theta}} = \theta + \int 2u\,du = \theta + \cancel{2}\cdot\frac{u^2}{\cancel{2}} + C = \theta + \sin^2\theta + C\,.$$

Remark: It is possible that you did it a different way, and came up with

$$I = \theta - \cos^2\theta + C\,.$$

This is actually the same answer, because in the second case we can write

$$I = \theta + \left(-\cos^2\theta + 1\right) + (C-1) = \theta + \sin^2\theta + (C-1)\,.$$

Since C can be *any* constant, this is actually the same as the first answer. (A computer algebra system I use came up with the second answer, which is why I thought about this.)

(e) $\displaystyle\int_0^{1/3} \frac{1}{9t^2+1}\,dt$

We can integrate $1/(1+u^2)$, so let $u = 3t$. We have

$$\frac{du}{dt} = 3 \quad\Longrightarrow\quad dt = \frac{du}{3}\,.$$

Moreover, the limits of integration change from

$$x = 0 \text{ to } u = 3\cdot 0 = 0 \quad \text{and} \quad x = \frac{1}{3} \text{ to } u = 3\cdot\frac{1}{3} = 1\,.$$

Hence

$$\int_0^{\frac{1}{3}} \frac{1}{9t^2+1}\,dt = \int_0^1 \frac{1}{1+u^2}\cdot\frac{du}{3} = \frac{1}{3}\arctan u\Big|_0^1 = \frac{1}{3}(\arctan 1 - \arctan 0) = \frac{1}{3}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{6}\,.$$

(f) $\displaystyle\int \frac{3t}{9t^2+1}\,dt$

One small change makes a big difference! We can integrate $1/u$, so let $u = 9t^2+1$. We have

$$\frac{du}{dt} = 18t \quad\Longrightarrow\quad dt = \frac{du}{18}\,.$$

Hence

$$\int \frac{3t}{9t^2+1}\,dt \quad=\quad \int \frac{3t}{u}\cdot\frac{du}{18t} \quad=\quad \frac{1}{6}\int\frac{1}{u}\,du \quad=\quad \frac{1}{6}\ln|u|+C \quad=\quad \frac{1}{6}\ln\left(9t^2+1\right)+C\,.$$

(We don't need an absolute value because $9t^2+1>0$ for all t.)

(f) $\displaystyle\frac{d}{dx}\int_0^{2x} e^{t^2}\,dt$

This is a *derivative* of an integral, not just an integral. You can't actually compute this integral, in fact! So we have to use the Fundamental Theorem of Calculus, Part II:

$$\frac{d}{dx}\int_0^{2x} e^{t^2}\,dt \quad=\quad e^{(2x)^2}\underbrace{\cdot 2}_{\text{Chain}}\,.$$

Do *only one* of questions 3 or 4.

3. Find the region between the curves $f(x) = 9 - x^2$ and $g(x) = x - 3$.
No one seems to have noticed the typo: "Find the *area of the* region between the curves…", but that is what was meant. To do this, first find the intersections:

$$9 - x^2 = x - 3 \quad\Longrightarrow\quad 0 = x^2 + x - 12 \quad\Longrightarrow\quad 0 = (x+4)(x-3) \quad\Longrightarrow\quad x = 3, -4\,.$$

The area between the curves is thus

$$\begin{aligned}\int_{-4}^{3} \left(9 - x^2\right) - (x - 3)\ dx &= \int_{-4}^{3} -x^2 - x + 12\, dx\\ &= \left(-\frac{x^3}{3} - \frac{x^2}{2} + 12x\right)\Big|_{-4}^{3}\\ &= \left(-\frac{27}{3} - \frac{9}{2} + 12(3)\right) - \left(-\frac{-64}{3} - \frac{16}{2} + 12(-4)\right)\\ &= -\frac{91}{3} + \frac{7}{2} + 12\cdot 7\\ &= \frac{343}{6}\,.\end{aligned}$$

4. If the marginal cost of producing b hamburgers at your local Burger Mac is $MC(b) = e^{b/4} - 10b + 100$ dollars, find the net change in cost of increasing production from 10 to 20 hamburgers. Round to the nearest cent.
Marginal cost is the cost of producing one additional unit. It is approximately equal to the derivative. Since net cost is the accumulation of the derivative of cost, we can approximate it using the accumulation of marginal cost. In other words, we can compute net cost using the integral of marginal cost:

$$\begin{aligned}C(20) - C(10) &= \int_{10}^{20} MC(b)\ db\\ &= \int_{10}^{20} e^{\frac{b}{4}} - 10b + 100\, db\\ &= \left(4e^{\frac{b}{4}} - 5b^2 + 100b\right)\Big|_{10}^{20}\\ &= \left[\left(4e^5 - 5\cdot 400 + 100\cdot 20\right) - \left(4e^{2.5} - 5\cdot 100 + 100\cdot 10\right)\right]\\ &\approx 44.92.\end{aligned}$$

So the net change in cost is approximately \$44.92.

6. There are two Mean Value Theorems in Calculus.
   (a) Give a precise statement of the Mean Value Theorem for *Integrals.*
   If the function f is continuous on $[a,b]$, then we can find $c \in (a,b)$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\ dx\ .$$

You can also say, "…such that

$$f(c) \cdot (b-a) = \int_a^b f(x)\ dx\ .$$

   (b) Give a geometric description of the Mean Value Theorem for *Integrals.*
   If the function f is unbroken on $[a,b]$, then we can find an x-value in (a,b) whose y-value is the average value of f on $[a,b]$.
   You can also say, "…we can find an x-value $c \in (a,b)$ such that the area under f is the same as a rectangle whose width is c's y-value and whose length is $b-a$.
   (c) Suppose $f(x) = \cos(\pi x/2)$. Find a point $c \in (1,3)$ such that $f(c)$ is the average value of f on $[1,3]$.
   We basically have to apply MVTI. We want some $c \in (1,3)$ such that

$$f(c) = \frac{1}{3-1} \int_1^3 \cos\left(\frac{\pi x}{2}\right)\ .$$

We already computed this integral! See 1(c). So we want some c such that

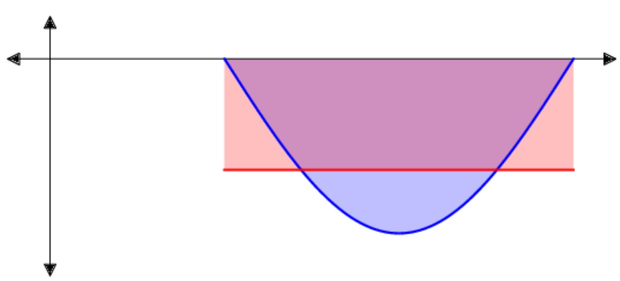
$$\cos\left(\frac{\pi c}{2}\right) = \frac{1}{2} \cdot \underbrace{\left(-\frac{4}{\pi}\right)}_{\text{see 1(c)}}$$

$$\cos\left(\frac{\pi c}{2}\right) = -\frac{2}{\pi}$$

$$\frac{\pi c}{2} = \cos^{-1}\left(-\frac{2}{\pi}\right)$$

$$c = \frac{2}{\pi}\cos^{-1}\left(-\frac{2}{\pi}\right)\ .$$

You can see the relationship when we plot the curve $f(x)$ and the line $y = f(c)$:



   (d) **(Bonus)** Give a geometric statement of the Mean Value Theorem for *Derivatives.*
   If the function f is unbroken on $[a,b]$ and smooth on (a,b), then we can find an x-value $c \in (a,b)$ such that the slope of the line tangent to f at c is the same as the slope of f's secant line on $[a,b]$.

(e) **(Bonus)** We used the Mean Value Theorem for *Derivatives* to prove the Fundamental Theorem of Calculus. However, we didn't use it *directly*, but rather *indirectly;* that is, we used one of its consequences. I have placed the proof on the back of this test [here, below the solution]. Indicate which line(s) use(s) the Mean Value Theorem for Derivatives indirectly. Line 4 is the line in question. To be precise, FTC Part I tells us that the derivative of A is f; we just said in (3) that F is an antiderivative of f, so A and F have the same derivative. The book's Theorem 4.11 tells us that any two functions with the same derivative differ only by a constant: this gives us the conclusion of Line 4, but Theorem 4.11 is a consequence of the Mean Value Theorem for Derivatives.

Fundamental Theorem of Calculus, Part II. *$\int_a^b f(x)\ dx = F(b) - F(a)$, where F is* any *antiderivative of f.*

Proof.

(1) For the sake of convenience write $A(x) = \int_a^x f(t)\ dt$.
(2) Notice that A is an antiderivative of f.
(3) Let F be *any* antiderivative of f.
(4) By FTC Part I, A and F differ only by a constant, C.
(5) Hence $F(a) = A(a) + C$ and $F(b) = A(b) + C$.
(6) By substitution, $F(b) - F(a) = [A(b) + C] - [A(a) + C]$.
(7) By cancellation, $F(b) - F(a) = \int_a^b f(t)\ dt - \int_a^a f(t)\ dt$.
(8) But $\int_a^a f(t)\ dt = 0$ because it has no actual area.
(9) Hence $F(b) - F(a) = \int_a^b f(t)\ dt$. □