TEST 1 FORM A (SOLUTIONS)

MAT 168

Directions: Solve as many problems as well as you can in the blue examination book, writing in pencil and showing all work. Put away any cell phones; the mere appearance will give a zero.

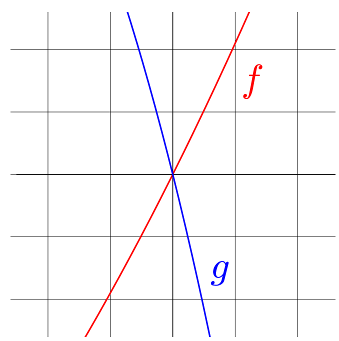
1. What does it mean for a function $f(x)$ to be differentiable on an interval $[a,b]$? Give both a geometric and an algebraic definition.
   geometric: the graph of f is smooth at every point in $[a,b]$ (no corners, cusps, kinks, or breaks)
   algebraic: the derivative of f exists at every point in $[a,b]$

2. Compute the following limits, if they exist. Use L'Hôpital's Rule *only* if necessary.

   (a) $\lim_{x\to 0} \frac{1}{x^2}$ (b) $\lim_{x\to\infty} \frac{e^{2x}}{x^2}$

   (c) $\lim_{x\to\frac{\pi}{2}} \frac{\tan 3x}{\sec 3x}$ (d) $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{5x}$

   (e) The figure at right shows two functions that intersect at $(0,0)$. Estimate
   $$\lim_{x\to 0} \frac{f(x)}{g(x)} .$$



(a) L'Hôpital's Rule does *not* apply: $\lim_{x\to 0^-} \frac{1}{x^2} = \infty$ and $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$ so $\lim_{x\to 0} \frac{1}{x^2} = \infty$

(b) L'Hôpital's Rule applies, multiple times:
$$\lim_{x\to\infty} \frac{e^{2x}}{x^2} \begin{matrix}\to \\ \to\end{matrix} \frac{\infty}{\infty} \quad \therefore \lim_{x\to\infty} \frac{e^{2x}}{x^2} = \lim_{x\to\infty} \frac{2e^{2x}}{2x} \begin{matrix}\to \\ \to\end{matrix} \frac{\infty}{\infty} \quad \lim_{x\to\infty} \frac{e^{2x}}{x^2} = \lim_{x\to\infty} \frac{4e^{2x}}{2} = \infty$$

(c) While L'Hôpital's Rule applies, it isn't very helpful:
$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 3x}{\sec 3x} \begin{matrix}\to \\ \to\end{matrix} \frac{\infty}{\infty} \quad \therefore \lim_{x\to\frac{\pi}{2}} \frac{\tan 3x}{\sec 3x} = \lim_{x\to\frac{\pi}{2}} \frac{\sec^2 3x}{\sec 3x \tan 3x} = \lim_{x\to\frac{\pi}{2}} \frac{\sec 3x}{\tan 3x} .$$
Round and round we'd go. Here it's better to notice that you can use trigonometry:
$$\lim_{x\to\frac{\pi}{2}} \frac{\tan 3x}{\sec 3x} = \lim_{x\to\frac{\pi}{2}} \frac{\sin 3x/\cos 3x}{1/\cos 3x} = \lim_{x\to\frac{\pi}{2}} \sin 3x = \sin\frac{3\pi}{2} = -1 .$$

(d) L'Hôpital's Rule applies indirectly, because
$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{5x} \to 1^\infty .$$

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To handle these cases we need to move the variable out of the exponent. That requires a logarithm:

$$\begin{aligned} y &= \left(1+\frac{3}{x}\right)^{5x} \\ \ln y &= \ln\left(1+\frac{3}{x}\right)^{5x} \\ \ln y &= 5x\ln\left(1+\frac{3}{x}\right) . \end{aligned}$$

Now we have

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\left[5x\ln\left(1+\frac{3}{x}\right)\right] \to \infty\cdot 0 .$$

This requires us to rewrite as a fraction:

$$\lim_{x\to\infty}\left[5x\ln\left(1+\frac{3}{x}\right)\right] = \lim_{x\to\infty}\frac{\ln\left(1+\frac{3}{x}\right)}{1/5x}\begin{matrix}\to\\ \to\end{matrix}\frac{0}{0} .$$

Now we can apply L'Hôpital's Rule:

$$\lim_{x\to\infty}\left[5x\ln\left(1+\frac{3}{x}\right)\right] = \lim_{x\to\infty}\frac{\ln\left(1+\frac{3}{x}\right)}{1/5x} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{3}{x}}\cdot\left(-\frac{3}{x^2}\right)}{-\frac{1}{5x^2}} = \lim_{x\to\infty}\frac{15}{1+\frac{3}{x}} = 15 .$$

Again, this is the limit of $\ln y$, not y itself. So

$$\lim_{x\to\infty}\ln y = 15 \quad\Longrightarrow\quad \lim_{x\to\infty} y = e^{15} .$$

(e) Since the functions intersect at the origin, we know that

$$\lim_{x\to 0}\frac{f(x)}{g(x)}\begin{matrix}\to\\ \to\end{matrix}\frac{0}{0} ,$$

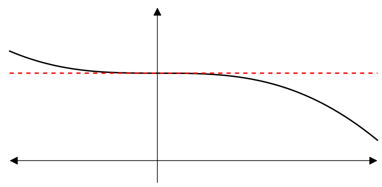
so L'Hôpital's Rule applies directly:

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} .$$

We know neither f' nor g', but we can find $f'(0)$ and $g'(0)$ because they are the slopes of the lines tangent to f and g at $x=0$. Draw the tangent lines for those curves at $(0,0)$ and it looks as if the tangent line for f has slope 2 while the tangent line for g has slope -6 (I'd be flexible in how close you come to these values). It seems that

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \frac{2}{-6} = -\frac{1}{3} .$$

3. Suppose we want to approximate a root of $y = 1 - x + \sin x$ using Newton's method.
   (a) Why would $x = 0$ be a catastrophic place to start? (*Hint:* Look at the graph of h.)
   At that point, $y' = 0$. Since Newton's method obtains new approximations from where the tangent line intersects the axis, and the tangent line at $x = 0$ never intersects the axis, this is quite the catastrophe:

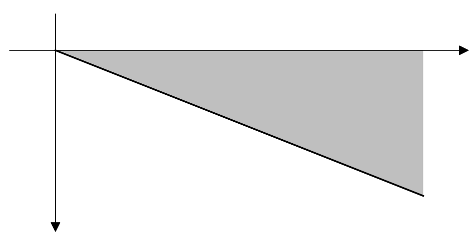


   (b) Suppose we start at $x = 1$. Find the next four approximations.
   Newton's method tells us that $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, and $y' = 0 - 1 + \cos x = \cos x - 1$, so

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 2.8305 \\ x_2 &= 2.0496 \\ x_3 &= 1.9387 \\ x_4 &= 1.9346 \end{aligned}$$

   (c) Is the last approximation correct to the nearest thousandth place? Why or why not?
   The thousandths digit is still changing, so we have no evidence that the last approximation is correct to the nearest thousandth place.

4. Use **geometry** to find $\int_0^b cx\ dx$, where b is positive but c is negative. Some words should explain the reasoning.
   The graph of $y = cx$ is a line of slope c. The region between the line and the x-axis forms a triangle:



   The integral is the area of the triangle, which is

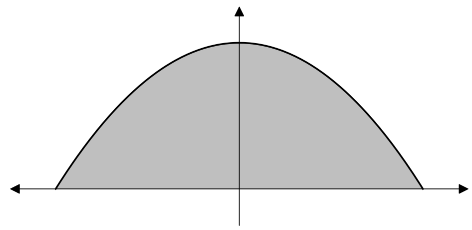
$$\int_0^b cx\,dx \quad = \quad \frac{1}{2} \cdot \underbrace{b}_{\text{base}} \cdot \underbrace{(cb)}_{\text{height}} = \frac{1}{2}cb^2\ .$$

   The area is negative because c is negative. This makes sense from the graph.

5. We want to find or approximate

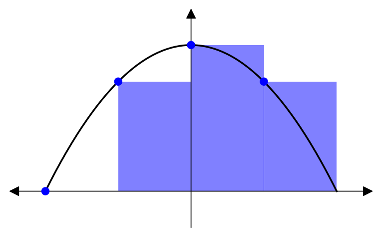
$$A = \int_{-1}^{1} 1 - x^2 \, dx \, .$$

(a) Sketch a graph of the region. Do we expect the area to be positive or negative?
The region looks like:



It is all above the x-axis, so the area is positive.

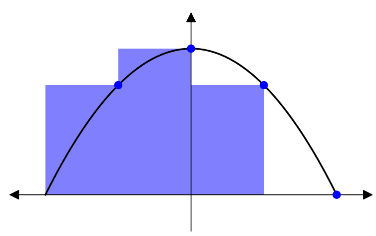
(b) Use four rectangles and left endpoints to approximate A.
Geometrically, our approximation looks like this:



The areas of the rectangles are determined by the left-hand endpoints of each subinterval. So

$$A \approx \frac{1}{2} \cdot f(-1) + \frac{1}{2} \cdot f\left(-\frac{1}{2}\right) + \frac{1}{2} \cdot f(0) + \frac{1}{2} \cdot f\left(\frac{1}{2}\right) = \frac{1}{2}\left[0 + \frac{3}{4} + 1 + \frac{3}{4}\right] = \frac{10}{8} = \frac{5}{4} \, .$$

(c) Use four rectangles and right endpoints to approximate A.
Geometrically, our approximation looks like this:



The areas of the rectangles are determined by the right-hand endpoints of each subinterval. So

$$A \approx \frac{1}{2} \cdot f\left(-\frac{1}{2}\right) + \frac{1}{2} \cdot f(0) + \frac{1}{2} \cdot f\left(\frac{1}{2}\right) + \frac{1}{2} \cdot f(1) = \frac{1}{2}\left[\frac{3}{4} + 1 + \frac{3}{4} + 0\right] = \frac{10}{8} = \frac{5}{4} \, .$$

(d) Something interesting just happened. What was it, and why did it happen?
Both (b) and (c) have the same value. This is due to the symmetry of the curve.

(e) Use the **definition of the integral** to find the exact value of A.

We have $a=-1$, $b=1$, $\Delta x = {}^{(1-(-1))}/_{n} = {}^{2}/_{n}$. It's easiest to use left endpoints, which gives us $x_i = -1 + i \cdot {}^{2}/_{n} = -1 + {}^{2i}/_{n}$. Put this into the definition and we have

$$\begin{aligned}
\int_0^1 1 - x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 - x_i^2\right) \cdot \frac{2}{n} \\
&= \lim_{n\to\infty} \left[\frac{2}{n} \sum_{i=1}^{n} \left[1 - \left(-1 + \frac{2i}{n}\right)^2\right]\right] \\
&= \lim_{n\to\infty} \left[\frac{2}{n} \sum_{i=1}^{n} \left[1 - \left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right)\right]\right] \\
&= \lim_{n\to\infty} \left[\frac{2}{n} \sum_{i=1}^{n} \left(\frac{4i}{n} - \frac{4i^2}{n^2}\right)\right] \\
&= \lim_{n\to\infty} \left[\frac{2}{n} \left(\sum_{i=1}^{n} \frac{4i}{n} - \sum_{i=1}^{n} \frac{4i^2}{n^2}\right)\right] \\
&= \lim_{n\to\infty} \left[\frac{2}{n} \left(\frac{4}{n}\sum_{i=1}^{n} i - \frac{4}{n^2}\sum_{i=1}^{n} i^2\right)\right] \\
&= \lim_{n\to\infty} \left[\frac{2}{n} \left(\frac{4}{n} \cdot \frac{n(n+1)}{2} - \frac{4}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right)\right] \\
&= \lim_{n\to\infty} \left[\frac{2}{n} \left(2(n+1) - \frac{2(n+1)(2n+1)}{3n}\right)\right] \\
&= \lim_{n\to\infty} \left[\frac{4(n+1)}{n} - \frac{4(n+1)(2n+1)}{3n^2}\right] \\
&= \lim_{n\to\infty} \left(\frac{4n+4}{n} - \frac{8n^2+12n+4}{3n^2}\right) \\
&= 4 - \frac{8}{3} \\
&= \frac{4}{3}\,.
\end{aligned}$$

6. **(bonus)** Explain how estimating the area under a curve when you know $f(x)$ is related to estimating the distance traveled when you know velocity $v(t)$.
To estimate the area under a curve, we break the interval into subintervals and assume the curve is a constant height on each subinterval, giving us rectangles. By adding the rectangles' areas we obtain an approximation of the area under the curve.

To estimate the distance traveled when we know velocity, we break the interval into subintervals and assume the velocity is constant on each subinterval, where we can use $D = RT$. By adding the distances traveled we obtain an approximation of the total distance traveled.

Useful formulas

Left endpoints: $x_i^* = a + (i-1)\,\Delta x$
Right endpoints: $x_i^* = a + i\Delta x$
Midpoints: $x_i^* = a + \left(i - \frac{1}{2}\right)\Delta x$
Sum shortcuts:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{i\,(i+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n\,(n+1)\,(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2\,(n+1)^2}{4}$$