TEST 1 FORM A (SOLUTIONS)

MAT 168

Directions: Solve as many problems as well as you can in the blue examination book, writing in pencil and showing all work. Put away any cell phones; the mere appearance will give a zero.

1. Give both a geometric and an algebraic definition of $\int_a^b f(x)\ dx$.

   geometric: the net area between $f(x)$ and the x-axis on the interval $[a, b]$

   algebraic: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$ where x_i^* is any sample point in the ith subinterval of $[a, b]$ and $\Delta x = \frac{b-a}{n}$

2. Compute the following limits, if they exist. Use L'Hôpital's Rule *only* if necessary.

   (a) $\lim_{x\to\infty} \frac{1}{x}$

   As $x \to \infty$, we are dividing a constant by larger and larger positive numbers, to the quotient dwindles to zero, so the limit is zero. L'Hôpital's Rule does not apply.

   (b) $\lim_{x\to 0^+} \frac{1}{x}$

   As $x \to 0$ from the right, we are dividing a constant by smaller and smaller positive numbers, so the quotient grows without bound, so the limit is ∞. L'Hôpital's Rule does not apply.

   (c) $\lim_{x\to 0} \frac{x - \sin 2x}{x}$

   $\lim_{x\to 0} \frac{x - \sin 2x}{x} \begin{matrix} \to \\ \to \end{matrix} \frac{0}{0}$, so L'Hôpital's Rule applies directly. Take the derivative of numerator and denominator and we have

   $$\lim_{x\to 0} \frac{x - \sin 2x}{x} \stackrel{\text{L'H}}{=} \lim_{x\to 0} \frac{1 - 2\cos 2x}{1} = \frac{1-2}{1} = -1\ .$$

   (d) $\lim_{x\to\infty} \frac{\ln(2x^2-3)}{\ln(x^3+1)}$

   $\lim_{x\to\infty} \frac{\ln(2x^2-3)}{\ln(x^3+1)} \begin{matrix} \to \\ \to \end{matrix} \frac{\infty}{\infty}$, so L'Hôpital's Rule applies directly. Take the derivative of numerator and denominator, *without forgetting the chain rule,* and we have

   $$\lim_{x\to\infty} \frac{\ln(2x^2-3)}{\ln(x^3+1)} \stackrel{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{4x}{2x^2-3}}{\frac{3x^2}{x^3+1}} = \lim_{x\to\infty} \frac{4x(x^3+1)}{3x^2(2x^2-3)} \begin{matrix} \to \\ \to \end{matrix} \frac{0}{0}\ ,$$

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so L'Hôpital's Rule applies *again*. We have

$$\lim_{x\to\infty} \frac{\ln(2x^2-3)}{\ln(x^3+1)} = \lim_{x\to\infty} \frac{4x^4+4x}{6x^4-9x^2} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{16x^3+4}{24x^3-18x} \overset{\text{L'H}}{=} \cdots = \frac{2}{3}\ .$$

(e) $\lim_{x\to\infty}\left(2x-\sqrt{4x^2-3x}\right)$

$\lim_{x\to\infty}\left(2x-\sqrt{4x^2-3x}\right) \to \infty - \infty$, so L'Hôpital's Rule, *if* it applies, applies indirectly. Multiply by the conjugate and we have

$$\lim_{x\to\infty} \frac{2x-\sqrt{4x^2-3x}}{1}\cdot\frac{2x+\sqrt{4x^2-3x}}{2x+\sqrt{4x^2-3x}} = \lim_{x\to\infty} \frac{4x^2-(4x^2-3x)}{2x+\sqrt{4x^2-3x}} = \lim_{x\to\infty} \frac{3x}{2x+\sqrt{4x^2-3x}} \begin{matrix}\to\\ \to\end{matrix} \frac{\infty}{\infty}\ .$$

Here L'Hôpital's Rule applies directly, and we have

$$\lim_{x\to\infty}\left(2x-\sqrt{4x^2-3x}\right) = \lim_{x\to\infty} \frac{3x}{2x+\sqrt{4x^2-3x}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{3}{2+\frac{8x-3}{2\sqrt{4x^2-3x}}}\ ,$$

which is a right mess, but multiply numerator and denominator by $\sqrt{4x^2-3x}$ and we have

$$\lim_{x\to\infty}\left(2x-\sqrt{4x^2-3x}\right) = \lim_{x\to\infty} \frac{3\sqrt{4x^2-3x}}{2\sqrt{4x^2-3x}+(8x-3)}\ .$$

This still approaches ∞/∞, and even worse, it seems to grow more complicated rather than less, so we contemplate a different approach: multiply numerator and denominator by $1/x$. That gives us

$$\lim_{x\to\infty}\left(2x-\sqrt{4x^2-3x}\right) = \lim_{x\to\infty} \frac{3\sqrt{4x^2-3x}}{2\sqrt{4x^2-3x}+(8x-3)}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{3\sqrt{4-\frac{3}{x}}}{2\sqrt{4-\frac{3}{x}}+\left(8-\frac{3}{x}\right)} = \frac{12}{16} = \frac{3}{4}\ .$$

(f) $\lim_{x\to 0^+}(1+3x)^{\frac{2}{x}}$

$\lim_{x\to 0^+}(1+3x)^{\frac{2}{x}}$ approaches the form 1^∞, so L'Hôpital's Rule, *if* it applies, applies indirectly. Since the variable is in the exponent, we need to use the natural logarithm:

$$\begin{aligned} y &= (1+3x)^{\frac{2}{x}} \\ \ln y &= \ln(1+3x)^{\frac{2}{x}} \\ \ln y &= \frac{2}{x}\cdot\ln(1+3x) \\ \ln y &= \frac{2\ln(1+3x)}{x} \\ \lim_{x\to 0^+}\ln y &= \lim_{x\to 0^+}\frac{2\ln(1+3x)}{x} \begin{matrix}\to\\ \to\end{matrix} \frac{0}{0}\ . \end{aligned}$$

Now L'Hôpital's Rule apples, and we have

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{2\ln(1+3x)}{x} \overset{\text{L'H}}{=} \lim_{x\to 0^+}\frac{2\cdot\frac{3}{1+3x}}{1} = \lim_{x\to 0^+}\frac{6}{1+3x} = 6\ .$$

This is the limit of $\ln y$, so

$$\lim_{x\to 0^+} (1+3x)^{\frac{2}{x}} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} = e^6 .$$

3. Suppose we want to approximate $\sqrt[3]{2}$ by using Newton's Method to find a root of $x^3 - 2$, starting at $x = 1$.
   (a) Find the first four approximations.
   We are given $f(x) = x^3 - 2$ and an initial approximation $x_0 = 1$. Newton's Method computes each successive approximation x_{i+1} using

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} ,$$

   so we compute $f'(x) = 3x^2$ and thus

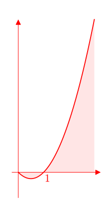
$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^3 - 2}{3 \cdot 1^2} = \frac{4}{3} \approx 1.3333 \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.3333 - \frac{1.3333^3 - 2}{3 \cdot 1.3333^2} \approx 1.2369 \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.2369 - \frac{1.2369^3 - 2}{3 \cdot 1.2369^2} \approx 1.2604 \\ x_4 &= x_4 - \frac{f(x_4)}{f'(x_4)} = 1.2604 - \frac{1.2604^3 - 2}{3 \cdot 1.2604^2} \approx 1.2599 \end{aligned}$$

   (b) Are they correct to the nearest thousandth place? Why or why not?
   The approximations are not guaranteed correct because they have not yet repeated up to the thousandths place. (Note: after this point they actually do start repeating, so technically they are correct, but you don't actually know that yet. I would not penalize you for saying "no, they're not accurate" when in fact they are but you lack evidence.)

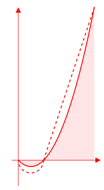
4. We want to find or approximate

$$A = \int_0^3 x^2 - x \, dx .$$

   (a) Use high school geometry to *approximate* the area "under" the curve. To be clear: I do not want anything sophisticated here. I do not expect you to find the exact area. Your grade depends on how intelligently you use ideas of high school geometry to approximate the area. As long as it makes sense, you earn full credit.
   The graph of $x^2 - x$ looks like this:



   It looks as if we could use a semicircle and a triangle to approximate the area beneath it:



The area of the semicircle would be

$$\frac{1}{2} \cdot \pi \cdot \left(\frac{1}{2}\right)^2 = \frac{\pi}{8} \ ,$$

while the area of the triangle would be

$$\frac{1}{2} \cdot 2 \cdot 6 = 6 \ .$$

Because the semicircle is *beneath* the x-axis, we subtract its area from that of the triangle, so we have

$$6 - \frac{\pi}{8} \approx 5.6 \ .$$

(b) Use three rectangles and left endpoints to approximate A.
We have $\Delta x = {}^{(3-0)}/_{3} = 1$. The left endpoints are thus $x = 0$, $x = 1$, and $x = 2$, whence

$$A \approx [f(0) + f(1) + f(2)] \cdot 1 = (0 + 0 + 2) \cdot 1 = 2 \ .$$

(c) Use six rectangles and left endpoints to approximate A.
We have $\Delta x = {}^{(3-0)}/_{6} = {}^{1}/_{2}$. The left endpoints are thus $x = 0$, $x = {}^{1}/_{2}$, $x = 1$, $x = {}^{3}/_{2}$, $x = 2$, and $x = {}^{5}/_{2}$, whence

$$\begin{aligned} A &\approx \left[f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right)\right] \cdot \frac{1}{2} \\ &= \left(0 - \frac{1}{4} + 0 + \frac{3}{4} + 2 + \frac{15}{4}\right) \cdot \frac{1}{2} \\ &= \frac{25}{8} \\ &= 3.125 \ . \end{aligned}$$

(d) Use the definition of the integral to find the exact value of A.

We have $\Delta x = {}^{(3-0)}/_{n}$ and, since we use right endpoints, $x_i^* = 0 + i \cdot {}^{3}/_{n} = {}^{3i}/_{n}$. Thus

$$\begin{aligned}
\int_0^3 x^2 - x \, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(x_i^*\right)^2 - x_i^*\right]\Delta x \\
&= \lim_{x\to\infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - \frac{3i}{n}\right] \cdot \frac{3}{n} \\
&= \lim_{x\to\infty} \frac{3}{n}\left[\frac{9}{n^2}\sum_{i=1}^{n} i^2 - \frac{3}{n}\sum_{i=1}^{n} i\right] \\
&= \lim_{x\to\infty} \frac{3}{n}\left[\frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{3}{n}\cdot\frac{n(n+1)}{2}\right] \\
&= \lim_{x\to\infty} \left[\frac{27n(n+1)(2n+1)}{6n^3} - \frac{9n(n+1)}{2n}\right] \\
&\vdots \quad \text{(using L'Hôpital's Rule or some other method)} \\
&= \frac{27\cdot 2}{6} - \frac{9}{2} \\
&= \frac{9}{2}\,.
\end{aligned}$$

(Our approximation in part (a) wasn't so bad, after all!)

(e) Why do we expect (c) to be more accurate than (b), and (d) to be exact?

We expect (c) to be more accurate because it uses more rectangles, and thus (typically) leaves less room for error. We expect (d) to be exact because, as the number of rectangles approaches ∞, the error should correspondingly approach 0.

5. **(bonus)** Suppose we know that A_L is the approximation of $\int_a^b f(x)\ dx$ using n left endpoints, and we also want to find A_R, the approximation using n right endpoints. Use the geometry of these approximations to explain how we can find A_R by subtracting one easily-found value (which one?) and adding one easily-found value (which one?).

We can find A_R by removing the leftmost rectangle's area and adding the rightmost rectangle's area, because the left endpoint of one rectangle is the right endpoint of the rectangle to its immediate left. Hence

$$A_R = A_L - \underbrace{f(x_0)\Delta x}_{\text{leftmost rectangle}} + \underbrace{f(x_n)\Delta x}_{\text{rightmost rectangle}}\ .$$

Useful formulas

Left endpoints: $x_i^* = a + (i-1)\,\Delta x$
Right endpoints: $x_i^* = a + i\Delta x$
Midpoints: $x_i^* = a + \left(i - \frac{1}{2}\right)\Delta x$
Sum shortcuts:

$$\sum_{i=1}^{n} c = cn \quad \sum_{i=1}^{n} i = \frac{i\,(i+1)}{2} \quad \sum_{i=1}^{n} i^2 = \frac{n\,(n+1)\,(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2\,(n+1)^2}{4}$$