DEFINITIONS/BIG-TIME FACTS TO KNOW FOR TEST 3

DEFINITIONS

All previous definitions, especially:

Applications of integrals: If the value V of a real-world phenomenon can be approximated by dividing an interval $[a,b]$ into n subintervals and adding the values of another function f on those subintervals, so that $V \approx \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$, then we can eliminate error by taking the limit as $n \to \infty$, so that the value of V is $\int_a^b f(x)\,dx$.

You should be able to use this principle to explain any application. For instance, were I to ask, "Why is $A = \int_a^b f(x)\,dx$ the formula for the area under a function f?" the best answer would be,

> We can approximate the area under f on $[a,b]$ by dividing $[a,b]$ into n subintervals and adding the areas of rectangles whose width is $\Delta x = (b-a)/n$ and whose height is $f\left(x_i^*\right)$, where x_i^* is a sample point on each subinterval. That is,
>
> $$A \approx \sum_{i=1}^{n} \ell_i w_i = \sum_{i=1}^{n} f(x_i^*)\Delta x \,.$$
>
> To eliminate the error, let the number of rectangles approach ∞, so that
>
> $$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\;dx \,.$$

You wouldn't need such a detailed answer for full credit; the main point is that I see:

- division of an interval;
- a reasonable explanation of the function used to approximate the quantity; and
- a reference to the limit giving us the integral.

Naturally, the more detailed and correct your answer is, the more likely you receive full credit.

Be ready to apply this principle to *any* of the following applications:

area between two curves	average value of a function	net change of a function
volume by slicing	volume by discs or washers	volume by shells

But also:

(definite) integral of $f(x)$ over $[a,b]$

(geometric) the *net* area between the curve of $f(x)$ and the x-axis, starting at $x=a$ and ending at $x=b$

(algebraic) $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$, where $\Delta x = b-a/n$ and x_i^* is *any* point in the ith subinterval of width Δx of $[a,b]$, *as long as the limit exists*

antiderivative of $f(x)$: any function F such that $F'(x) = f(x)$

(indefinite) integral of $f(x)$: an antiderivative of f

BIG-TIME RESULTS

You need not know the proofs. I include them only for the students who find it interesting.

Theorem (Pythagorean identities for trigonometric functions)**.** *For any angle* θ*,*

$$\begin{aligned}\sin^2\theta+\cos^2\theta&=1\\ \tan^2\theta+1&=\sec^2\theta\\ \tan^2\theta&=\sec^2\theta-1\ .\end{aligned}$$

Proof. The definitions of $\sin\theta$ and $\cos\theta$ from the right triangle are that $\sin\theta={}^{\text{opp}}/_{\text{hyp}}$ and $\cos\theta={}^{\text{adj}}/_{\text{hyp}}$ (SOHCAHTOA). From this we have

$$\sin^2\theta+\cos^2\theta=\left(\frac{\text{opp}}{\text{hyp}}\right)^2+\left(\frac{\text{adj}}{\text{hyp}}\right)^2=\frac{\text{opp}^2+\text{adj}^2}{\text{hyp}^2}\underset{\text{Pyth}}{=}\frac{\text{hyp}^2}{\text{hyp}^2}=1\ .$$

Having established that, we can divide both sides by $\cos^2\theta$ to obtain

$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}\quad\underset{\text{trig. def.}}{\Longrightarrow}\quad\tan^2\theta+1=\sec^2\theta\ .$$

We obtain the third identity by simply rewriting the second. □

Theorem (The half-angle formulas)**.** *For any angle* θ*,*

$$\sin^2\theta=\frac{1-\cos2\theta}{2}\quad\textit{and}\quad\cos^2\theta=\frac{1+\cos2\theta}{2}\ .$$

Proof. The double-angle formula for cosine tells us that

$$\cos2\theta=\cos^2\theta-\sin^2\theta\ . \tag{0.1}$$

The Pythagorean identity tells us that

$$\sin^2\theta+\cos^2\theta=1\quad\Longrightarrow\quad\sin^2\theta=1-\cos^2\theta\ . \tag{0.2}$$

Substitution equation (0.2) into equation (0.1) and we have

$$\cos2\theta=\cos^2\theta-\left(1-\cos^2\theta\right)\quad\Longrightarrow\quad\cos2\theta=2\cos^2\theta+1\ .$$

Isolate $\cos^2\theta$ and you have the second half-angle formula.

If we solve (0.2) for $\cos^2\theta$, a similar substitution gives the first half-angle formula. □

Theorem (The Chain Rule for Integrals; or, u-substitution)**.** *If f is a function of a variable u, which in turn is a function of x, and F is an antiderivative of f, then*

$$F(u)=\int f(u)\,u'\,dx=\int f(u)\,du.$$

Proof omitted, but available on request.

Theorem (Integration by parts)**.**

$$\int uv'\,dx \quad = \quad uv - \int u'v\,dx.$$

Proof. The product rule for derivatives states that

$$\frac{d}{dx}(uv) = u'v + uv'\,.$$

Integrate both sides,

$$\int \left[\frac{d}{dx}(uv)\right] dx \quad = \quad \int u'v + uv'\,dx$$

and by properties of integrals we have

$$uv \quad = \quad \int u'v\,dx + \int uv'\,dx\,.$$

Rewrite as

$$uv - \int u'v\,dx \quad = \quad \int uv'\,dx$$

and we are done. □

INTEGRATION TABLE

You must know them, as I will not provide these on the test. Don't leave off "$+C$" as I did.

$\int k\,dx = kx$	(k is a constant)
$\int x^n\,dx = \frac{x^{n+1}}{n+1}$	($n \neq 1$)
$\int \frac{1}{x}\,dx = \ln\lvert x\rvert$	
$\int e^x\,dx = e^x$	
$\int a^x\,dx = \frac{a^x}{\ln a}$	($a > 0$ but $a \neq 1$)
$\int \sin x\,dx = -\cos x$	$\int \cos x\,dx = \sin x$
$\int \sec^2 x\,dx = \tan x$	$\int \sec x \tan x\,dx = \sec x$
$\int \csc^2 x\,dx = -\cot x$	$\int \csc x \cot x\,dx = -\csc x$
$\int \frac{1}{1+x^2}\,dx = \arctan x$	
$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x$	
$\int \frac{1}{x\sqrt{x^2-1}} = \operatorname{arcsec} x$	

Integration Heuristic.

Table	Is the integrand in the table?
Geometry	Does the integral represent an easily-found geometric value? Can I take advantage of symmetry?
Algebra	Can I rewrite the integral by expanding or simplifying?
Substitution	Does the integrand have one function "inside" another? Do I see the inner function's derivative on the "outside"?
Parts	Does the integral look like a product? If so, and its two "parts" are uv', does u "reduce" when differentiating, and is v' "easier" to integrate? (The second question is a guideline, not a hard-and-fast rule!)
Trigonometry	Do I see $\sin^m \alpha \cos^n \alpha$ or $\tan^m \alpha \sec^n \alpha$?
Trig. subst.	Do I see a^2+x^2, a^2-x^2, or $\sqrt{x^2-a^2}$?

Example problems

This list is by no means exhaustive.

1. Simplify the following integrals.

(a) $\int \sin^2 2x\,dx$ (b) $\int x \sin 3x\,dx$ (c) $\int \ln(4x)\,dx$

(d) $\int 3x^2 \ln(2x)\,dx$ (e) $\int \cos(2x)\ln(\sin 2x)\,dx$ (f) $\int e^{-2x}\cos x\,dx$

(g) $\int \sin^4 x \cos^3 x\,dx$ (h) $\int \sin^4 x \cos^2 2x\,dx$ (i) $\int \sec^4 x\,dx$

(j) $\int_0^{\pi/4} \tan^5 x \sec x\,dx$ (k) $\int_{-\pi/4}^{\pi/4} \tan^3 x\,dx$ (l) $\int_{-4}^{4} \sqrt{16-x^2}\,dx$

(m) $\int_1^3 \sqrt{16-x^2}\,dx$ (n) $\displaystyle\int \frac{x^2}{\sqrt{x^2-16}}\,dx$ (o) $\int x^3\sqrt{x^2+9}\,dx$

2. Set up integrals for the following problems, but do not simplify them.
   (a) Find the average value of $f(x)=1/(3x)$ on the interval $[1,3]$. Diagram f and its average value in appropriate fashion.
   (b) If the marginal cost of producing n thousand widgets is $MC(n)=(n-3)^2-1$ thousands of dollars, determine the net change in cost to increase production from 1 thousand to 3 thousand widgets.
   (c) Find the total area of the region between the curves $f(x)=4-x$ and $g(x)=\sqrt{1-x^2}$ over the interval $[0,1]$. To evaluate the integral it may help to use geometry. Draw a diagram of the region.
   (d) Use the method of slicing to find the volume of the solid whose base is the region between $y=x^2$, the x-axis, and the line $x=1$ and whose cross-sections perpendicular to the x-axis are squares.
   (e) Use the method of disks and washers to find the volume of the solid formed by rotating the region defined in part (c) about the x-axis.
   (f) Use the method of shells to find the volume of the solid formed by rotating the region between $y=\sin(\pi x)$, the x-axis, $x=0$, and $x=1$ about the y-axis.
   (g) Use the method of shells to find the volume of the solid formed by rotating the region defined in part (c) about the y-axis.

Solutions to example problems

1.

(a) $\int \sin^2 x\, dx$

This integral doesn't appear on the table. It looks a little like both $\int \sin u\, du$ and $\int u^2\, du$, neither of which helps. You need instead to make use of the half-angle formula:

$$\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}\int 1-\cos 2x\, dx\ .$$

Integrating 1 is trivial, but integrating $\cos 2x$ requires substitution. Let $u = 2x$ and we have

$$\frac{du}{dx} = 2 \quad \Longrightarrow \quad \frac{du}{2} = dx\ .$$

Hence

$$\frac{1}{2}\int 1-\cos 2x\, dx = \frac{1}{2}\int (1-\cos u)\cdot\frac{du}{2} = \left(\frac{1}{2}\cdot\frac{1}{2}\right)\int 1-\cos u\, du$$
$$= \frac{1}{4}(u-\sin u)+C = \frac{1}{4}(2x-\sin 2x)+C\ .$$

(b) $\int x\sin 3x\, dx$

This integral looks like a product of x and $\sin 3x$, which is a clue that you should pursue integration by parts. (Besides, nothing else works.) The derivative of x is "simpler" than x, and the integral of $\sin 3x$ is "easy" to compute, so we set

$$u = x \quad v' = \sin 3x$$
$$u' = 1 \quad v = -\frac{1}{3}\cos 3x$$

(You can compute v using a u-substitution with $u = 3x$.) The integration by parts formula tells us that

$$\int uv'\, dx = uv - \int u'v\, dx = -\frac{1}{3}x\cos 3x - \int 1\cdot\left(-\frac{1}{3}\cos 3x\right) dx\ .$$

The second integral can be solved with a simple u-substitution ($u = 3x$ again) so the original integral simplifies to

$$I = -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\, dx = \frac{1}{3}\left[-x\cos 3x + \frac{1}{3}\sin 3x\right]+C.$$

(c) $\int \ln(4x)\, dx$

Although this doesn't look like a product, the only way to attack this is by parts. (This is common with expressions involving ln, though by no means universal.) The only choice to make is

$$u = \ln(4x) \quad v' = 1$$
$$u' = \frac{1}{4x}\underbrace{\cdot 4}_{\text{chain}} \quad v = x$$

(because $\ln(4x) = \ln(4x) \times 1$, so if you choose $u = \ln(4x)$ you have no choice for v' but 1). Notice u' simplifies to $1/x$. The integration by parts formula tells us that

$$\int uv'\,dx = uv - \int u'v\,dx = x\ln(4x) - \int \frac{1}{x}\cdot x\,dx = x\ln(4x) - \int dx = x\ln(4x) - x + C.$$

(d) $\int 3x^2 \ln(2x)\,dx$

This problems also requires us to integrate by parts. Choose

$$u = \ln(2x) \quad v' = 3x^2$$

$$u' = \frac{1}{\cancel{2}x}\cdot\cancel{2} \qquad v = \cancel{3}\times\frac{x^3}{\cancel{3}}$$

(unless you feel like integrating $\ln(2x)$, which can be done similarly to part (c)). The integration by parts formula tells us that

$$\int uv'\,dx = uv - \int u'v\,dx = x^3\ln(2x) - \int \frac{1}{x}\cdot x^3\,dx = x^3\ln(2x) - \int x^2\,dx = x^3\ln(2x) - \frac{x^3}{3} + C.$$

(e) $\int \cos(2x)\ln(\sin 2x)\,dx$

Although this looks like a product, and you *can* solve it using integration by parts, you can start with u-substitution. Let

$$u = \sin 2x \quad\Longrightarrow\quad \frac{du}{dx} = 2\cos 2x \quad\Longrightarrow\quad \frac{du}{2\cos 2x} = dx\ .$$

We can rewrite the integral as

$$\int \cancel{\cos(2x)}\ln u\,\frac{du}{2\cancel{\cos 2x}} = \frac{1}{2}\int \ln u\,du\ . \tag{0.3}$$

At this point you may recognize that our integral looks a lot like the one in (b), where we have to use integration by parts. Ordinarily we'd use u and v, but we've already used u once, so we'll use w and v instead. (Get it? w is "double u." Hyuck hyuck.) We have

$$w = \ln u \quad v' = 1$$

$$w' = \frac{1}{u} \qquad v = u$$

The integration by parts formula tells us that

$$\int wv'\,du = wv - \int w'v\,du = u\ln u - \int \frac{1}{u}\cdot u\,du = u\ln u - \int du = u\ln u - u + C\ .$$

Once we substitute back in for u, we get

$$\sin 2x \ln\sin 2x - \sin 2x + C\ .$$

However, we had a $1/2$ way up in equation (0.3) that I omitted while working on $\int \ln u\,du$; we have to add that back in. So the correct answer is

$$\frac{1}{2}\sin 2x \ln\sin 2x - \sin 2x + C\ .$$

On the other hand, suppose you decided to solve it by parts from the start. The best idea is to use

$$u = \ln(\sin 2x) \qquad v' = \cos(2x)$$
$$u' = \frac{1}{\sin 2x} \cdot \underbrace{2\cos 2x}_{\text{chain}} \qquad v = \frac{1}{2}\sin 2x$$

The integration by parts formula tells us that

$$\begin{aligned}\int uv'\,dx = uv - \int u'v\,dx &= \frac{1}{2}\sin 2x \ln(\sin 2x) - \int \frac{2\cos 2x}{\cancel{\sin 2x}} \cdot \frac{1}{2}\cancel{\sin 2x}\,dx \\ &= \frac{1}{2}\sin 2x \ln(\sin 2x) - \int \cos 2x\,dx \\ &= \frac{1}{2}\sin 2x \ln(\sin 2x) - \frac{1}{2}\sin 2x + C\,.\end{aligned}$$

We get the same answer, and this seems a little quicker, actually!

(f) $\int e^{-2x}\cos x\,dx$

This looks like a product, so we try integration by parts. Let

$$u = e^{-2x} \qquad v' = \cos x$$
$$u' = -2e^{-2x} \qquad v = \sin x$$

(You *could* swap the choices of u and v' and still get the right answer, but these are my solutions, so I'll do them how I want.) The integration by parts formula tells us that

(0.4)
$$I = \int uv'\,dx = uv - \int u'v\,dx = e^{-2x}\sin x - \int \left(-2e^{-2x}\right)\sin x\,dx = e^{-2x}\sin x + 2\int e^{-2x}\sin x\,dx\,.$$

We have a new integral that looks suspiciously similar to the one we started with, but it's different enough that we don't panic yet. Let

$$u = e^{-2x} \qquad v' = \sin x$$
$$u' = -2e^{-2x} \qquad v = -\cos x$$

(You *may not* at this point swap the choices of u and v'; that would be catastrophic. However, if you swapped the choices in the first step, then you should imitate your choice in the second step. If you don't understand what I'm saying in this aside, then ignore it: what matters is that you understand what's outside this parenthetical remark.) The integration by parts formula tells us that

(0.5)
$$\int uv'\,dx = uv - \int u'v\,dx = e^{-2x}(-\cos x) - \int \left(-2e^{-2x}\right)(-\cos x)\,dx = -e^{-2x}\cos x - 2\int e^{-2x}\cos x\,dx\,.$$

We now have an integral that looks exactly like the one we started with.

PANIC!

That's right; go ahead and get it out of your system. Just don't take too long, because you only have 50 minutes for the test. How, then, should you handle this problem? Put together the original integral I and equations (0.4) and (0.5) to see that

$$I = e^{-2x}\sin x + 2\left[-e^{-2x}\cos x - 2I\right]$$

which we can rewrite as

$$I = e^{-2x}\sin x - 2e^{-2x}\cos x - 4I \ .$$

We can actually solve for I by adding $4I$ to both sides:

$$5I = e^{-2x}\sin x - 2e^{-2x}\cos x \quad \Longrightarrow \quad I = \frac{1}{5}\left(e^{-2x}\sin x - 2e^{-2x}\cos x\right) \ .$$

(g) $\int \sin^4 x \cos^3 x \, dx$
It looks like a product, but it's a product of trigonometric functions, so we have to apply properties of trig functions. In this case, the power on cosine is odd, so rewrite all the cosines as sines *except one:*

$$I = \int \sin^4 x \cdot \cos^2 x \cdot \cos x \, dx \underbrace{=}_{\text{Pyth.}} \int \sin^4 x \left(1 - \sin^2 x\right) \cos x \, dx \ .$$

Now let $u = \sin x$; we have $du/dx = \cos x$, so $dx = du/\cos x$. Hence

$$I = \int u^4 \left(1 - u^2\right) \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}} = \int u^4 - u^6 \, du = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C \ .$$

(h) $\int \sin^4 x \cos^2 2x \, dx$
Again we need to apply properties of trig functions. This time, both powers are even, so we use half-angle formulas:

$$I = \int \left(\sin^2 x\right)^2 \cos^2 2x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)^2 \left(\frac{1 + \cos 4x}{2}\right) dx \ .$$

We have to expand the product:

$$\begin{aligned} I &= \frac{1}{8}\int \left(1 - 2\cos 2x + \cos^2 2x\right)\left(1 + \cos 4x\right) dx \\ &= \frac{1}{8}\int 1 + \cos 4x - 2\cos 2x - 2\cos 2x \cos 4x + \cos^2 2x + \cos^2 2x \cos 4x \ . \end{aligned}$$

The first three terms are relatively easy to integrate; u-substitution will do. The fifth term is a little harder, but like (a) a double-angle formula will do. So I will rewrite and separate I into two integrals,

$$I = \frac{1}{8}\int \underbrace{\left(1 + \cos 4x - 2\cos 2x + \cos^2 2x\right)}_{I_1} + \underbrace{\left(-2\cos 2x \cos 4x + \cos^2 2x \cos 4x\right)}_{I_2} .$$

Let's go ahead and dispose of I_1:

$$\begin{aligned} I_1 &= \int 1+\cos 4x - 2\cos 2x + \frac{1+\cos 4x}{2}\,dx \\ &= x + \frac{1}{4}\sin 4x - \sin 2x + \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right) \\ &= \frac{3}{2}x + \frac{3}{8}\sin 4x - \sin 2x\ . \end{aligned}$$

That leaves I_2, which mix $\cos 2x$ with $\cos 4x$. Use a double-angle formula to rewrite

$$\cos 4x = \left(1-2\sin^2 2x\right) = \left(2\cos^2 2x - 1\right)\ .$$

(Watch as we cleverly use both versions.) We now have, and split into two *new* integrals,

$$I_2 = \int \underbrace{\left(-2\cos 2x\left(1-2\sin^2 2x\right)\right)}_{I_3} + \underbrace{\left(\cos^2 2x\left(2\cos^2 2x-1\right)\right)}_{I_4} dx\ .$$

For I_3, a simple u-substitution will do. Let $u = \sin 2x$, then $du/dx = 2\cos 2x$, so $dx = du/2\cos 2x$ and

$$I_3 = \int -\cancel{2\cos 2x}\left(1-2u^2\right)\cdot\frac{du}{\cancel{2\cos 2x}} = -\int 1-2u^2\,du = -\sin 2x + \frac{2\sin^3 2x}{3}\ .$$

For I_4, we will have to use half-angle formulas again:

$$\begin{aligned} I_4 &= \int 2\cos^4 2x - \cos^2 2x\,dx \\ &= \int 2\left(\frac{1+\cos 4x}{2}\right)^2 - \frac{1+\cos 4x}{2}\,dx \\ &= \int \frac{1}{2}\left(1+2\cos 4x+\cos^2 4x\right) - \frac{1+\cos 4x}{2}\,dx\ . \end{aligned}$$

This simplifies to

$$I_4 = \int \frac{\cos 4x}{2} + \frac{\cos^2 4x}{2}\,dx = \int \frac{\cos 4x}{2} + \frac{1+\cos 8x}{4}\,dx\ .$$

(Notice we needed *yet another* half-angle formula.) So

$$I_4 = \frac{\sin 4x}{8} + \frac{1}{4}\left(x + \frac{1}{8}\sin 8x\right) = \frac{1}{8}\sin 4x + \frac{1}{4}\left(x+\frac{\sin 8x}{8}\right)\ .$$

Combining everything, including — let us not forget! — the $1/8$ that appeared in I, which we did not copy down into I_2, etc., we have

$$I = \frac{1}{8}\left[\underbrace{\frac{3}{2}x + \frac{3}{8}\sin 4x - \sin 2x}_{I_1} + \underbrace{\underbrace{\left(-\sin 2x + \frac{2\sin^3 2x}{3}\right)}_{I_3} + \underbrace{\frac{1}{8}\sin 4x + \frac{1}{4}\left(x+\frac{\sin 8x}{8}\right)}_{I_4}}_{I_2}\right]\ .$$

That looks like a lot, but... well, you're right. It is a lot. I wouldn't ask something quite this complicated on a test. In any case, I've checked it against a computer algebra system, though, and it works out.

(i) $\int \sec^4 x\, dx$
Since the power of secant is even, rewrite all but two secants as tangents using a Pythagorean identity:

$$I = \int \sec^2 x \cdot \sec^2 dx\, dx = \int \left(\tan^2 x + 1\right) \sec^2 x\, dx\ .$$

Let $u = \tan x$; then ${}^{du}/_{dx} = \sec^2 x$, so $dx = {}^{du}/_{\sec^2 x}$ and

$$I = \int \left(u^2 + 1\right) \cancel{\sec^2 x} \cdot \frac{du}{\cancel{\sec^2 x}} = \int u^2 + 1\, du = \frac{\tan^3 x}{3} + \tan x + C\ .$$

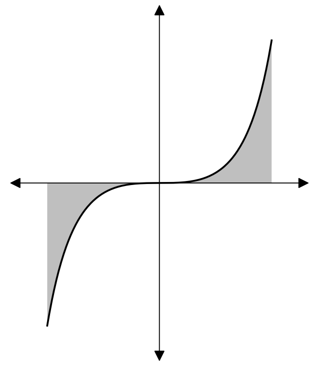
(j) $\int_0^{\pi/4} \tan^5 x \sec x\, dx$
Since the powers are both odd, isolate one secant and one tangent, and rewrite everything else as secants:

$$I = \int_0^{\pi/4} \tan^4 x \cdot \sec x \tan x\, dx = \int_0^{\pi/4} \left(\sec^2 x - 1\right)^2 \sec x \tan x\, dx\ .$$

Let $u = \sec x$; then ${}^{du}/_{dx} = \sec x \tan x$, so $dx = {}^{du}/_{(\sec x \tan x)}$ and

$$\begin{aligned}
I &= \int_1^{\sqrt{2}} \left(u^2 - 1\right)^2 \cancel{\sec x \tan x} \cdot \frac{du}{\cancel{\sec x \tan x}} \\
&= \int_1^{\sqrt{2}} u^4 - 2u^2 + 1\, du \\
&= \left(\frac{u^5}{5} - \frac{2u^3}{3} + u\right)\Bigg|_1^{\sqrt{2}} \\
&= \left(\frac{\sqrt{2}^5}{5} - \frac{2\sqrt{2}^3}{3} + \sqrt{2}\right) - \left(\frac{1}{5} - \frac{2}{3} + 1\right) \\
&= \frac{7\sqrt{2}}{15} - \frac{8}{15}\ .
\end{aligned}$$

(k) $\int_{-\pi/4}^{\pi/4} \tan^3 x\, dx$
If you merely consider the geometry, you see that the curve is symmetric, with half below the axis and half above. The net area is 0, so the integral is 0.



(l) $\int_{-4}^{4} \sqrt{16-x^2}\, dx$
Again, consider the geometry: $\sqrt{16-x^2}$ is the equation of a semicircle of radius 4 centered at the origin (from the equation of a circle $x^2+y^2=4^2$). The area is

$$\frac{\pi r^2}{2} = \frac{\pi \cdot 4^2}{2} = 8\pi.$$

The integral is the area, so $I = 8\pi$.

(m) $\int_1^3 \sqrt{16-x^2}\, dx$
Although this, too, is area under a semicircle, it isn't the entire area, so we can't apply the formula. In this case we have to use trigonometric substitution. We're looking at the form a^2-u^2, in which case we should substitute

$$x = a\sin\theta = 4\sin\theta \quad \Longrightarrow \quad \frac{dx}{d\theta} = 4\cos\theta \quad \Longrightarrow \quad dx = 4\cos\theta\, d\theta\,,$$

giving us

$$I = \int_{x=1}^{3} \sqrt{16-16\sin^2\theta} \cdot 4\cos\theta d\theta\,.$$

The point of this is to rewrite so that we can use the Pythagorean identity:

$$I = \int_{x=1}^{3} \sqrt{16}\sqrt{1-\sin^2\theta} \cdot 4\cos\theta\, d\theta = 16\int_{x=1}^{3} \sqrt{\cos^2\theta} \cdot \cos\theta\, d\theta = 16\int_{x=1}^{3} \cos^2\theta\, d\theta\,.$$

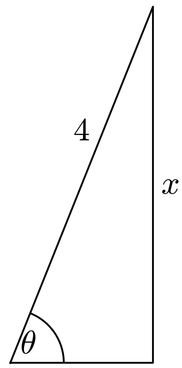
At this point we need to apply a half-angle formula:

$$I = 16\int_{x=1}^{3} \frac{1+\cos 2\theta}{2}\, d\theta = 8\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_{x=1}^{3}\,.$$

We have to rewrite in terms of x before substituting, but the second fraction has the form $\sin 2\theta$, so we also need to exploit a double-angle formula to end up in terms of just θ, rather than 2θ:

$$I = 8\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right)\Bigg|_{x=1}^{3} = 8\left(\theta + \sin\theta\cos\theta\right)\Big|_{x=1}^{3}\,.$$

Recall that $x = 4\sin\theta$, so $\sin\theta = x/4$ and $\theta = \arcsin(x/4)$. What about $\cos\theta$? We just pointed out that $\sin\theta = x/4$, so we investigate a right triangle where the leg opposite θ is x and the hypotenuse is 4:



In this case, $\cos\theta = {}^{\text{adj}}/_{\text{hyp}} = {}^{\sqrt{16-x^2}}/_{4}$. So the integral is

$$I = 8\left(\arcsin\left(\frac{x}{4}\right) + \frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4}\right)\Bigg|_1^3 = 8\left[\left(\arcsin\left(\frac{3}{4}\right) + \frac{3\sqrt{7}}{16}\right) - \left(\arcsin\left(\frac{1}{4}\right) + \frac{\sqrt{15}}{16}\right)\right].$$

(n) $\displaystyle\int \frac{x^2}{\sqrt{x^2-16}}\,dx$

Same issue, but this time we see $\sqrt{x^2-a^2}$ so we use

$$x = a\sec\theta \implies x = 4\sec\theta \implies \frac{dx}{d\theta} = 4\sec\theta\tan\theta\,,$$

so

$$\begin{aligned} I &= \int \frac{16\sec^2\theta}{\sqrt{16\sec^2\theta - 16}}\cdot 4\sec\theta\tan\theta\,d\theta \\ &= \int \frac{16\sec^2\theta}{\cancel{\sqrt{16}}\sqrt{\sec^2\theta-1}}\cdot\cancel{4}\sec\theta\tan\theta\,d\theta \\ &= \int \frac{16\sec^2\theta}{\cancel{\sqrt{\tan^2\theta}}}\cdot\sec\theta\cancel{\tan\theta}\,d\theta \\ &= 16\int \sec^3\theta\,d\theta\,. \end{aligned}$$

The power of the secant is odd, and there are no tangents, so the strategy is in circumstance is to

PRAY.

Once we have that out of the way, we use Reduction Formula #55 in the text's integration table, which states that

$$\int \sec^n(ax) = \frac{1}{a(n-1)}\sec^{n-2}(ax)\tan(ax) + \frac{n-2}{n-1}\int \sec^{n-2}(ax)\,dx\,.$$

In our case, which is $\sec^3\theta$, we have $n = 3$ and $a = 1$, so

$$\begin{aligned} I &= 16\left[\frac{1}{1\times(3-1)}\sec^{3-2}(1\theta)\tan(1\theta) + \frac{3-2}{3-1}\int \sec^{3-2}(1\theta)\,d\theta\right] \\ &= 16\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\,d\theta\right). \end{aligned}$$

For $\int \sec x\, dx$ you need another entry of the integration table, which number I forget offhand but it's

$$\int \sec x = \ln|\sec x + \tan x| \ ,$$

so that we have

$$I = 8(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) + C \ .$$

Now we have to return to the world of x's. Recall that $x = 4\sec\theta$, so $\sec\theta = x/4$. For $\tan\theta$ we use the same triangle idea as in the previous program to determine that $\tan\theta = \sqrt{x^2-16}/4$. That gives us

$$I = 8\left(\frac{x}{4}\cdot\frac{\sqrt{x^2-16}}{4} + \ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right|\right) + C = \frac{x\sqrt{x^2-16}}{2} + 8\ln\left|\frac{x+\sqrt{x^2-16}}{4}\right| + C \ .$$

[*Note:* While this is a good example of the *kind* of problem I could ask, it is not a good example of the *difficulty* I am aiming for on the test. This problem requires at least two entries from an integration table. I will not require you to know or use any entries from an integration table. I *might* ask you to prove a reduction formula, but it would be a simple one that proceeds by integration by parts, like one I did in class.]

(o) $\int x^3\sqrt{x^2+9}\, dx$

Same as before, but this time we see $x^2 + a^2$ so we use

$$x = a\tan\theta \implies x = 3\tan\theta \implies \frac{dx}{d\theta} = 3\sec^2\theta \ ,$$

so

$$\begin{aligned} I &= \int 27\tan^3\theta\cdot\sqrt{9\tan^2\theta+9}\cdot 3\sec^2\theta\, d\theta \\ &= 81\int \tan^3\theta\cdot\sqrt{9}\sqrt{\tan^2\theta+1}\cdot\sec^2\theta\, d\theta \\ &= 243\int \tan^3\theta\cdot\sqrt{\sec^2\theta}\cdot\sec^2\theta\, d\theta \\ &= 243\int \tan^3\theta\sec^3\theta\, d\theta \ . \end{aligned}$$

We dealt with something like this way back in (j). Both powers are odd, so we reserve one each of $\tan\theta$ and $\sec\theta$ and convert the remaining tangents to secants:

$$I = 243\int \tan^2\theta\sec^2\theta\cdot\tan\theta\sec\theta\, d\theta = 243\int \left(\sec^2\theta - 1\right)\sec^2\theta\cdot\sec\theta\tan\theta\, d\theta \ .$$

The whole point of this is to let $u = \sec\theta$, so that $du/d\theta = \sec\theta\tan\theta$, and

$$I = 243\int \left(u^2-1\right)u^2\cdot\cancel{\sec\theta\tan\theta}\cdot\frac{du}{\cancel{\sec\theta\tan\theta}} = 243\int u^4 - u^2\, du \ .$$

This is now easy:

$$I = 243\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C = 243\left(\frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3}\right) + C \ .$$

Once again, we have to convert back to the world of x's. Recall that $x = 3\tan\theta$, so $\tan\theta = x/3$. Unfortunately, we don't have any tangents in our answer; we have only secants. As before, build a right triangle to determine that $\sec\theta = \sqrt{x^2+9}/3$. Now we have

$$I = 243\left(\frac{\left(\frac{\sqrt{x^2+9}}{3}\right)^5}{5} - \frac{\left(\frac{\sqrt{x^2+9}}{3}\right)^3}{3}\right) + C$$

$$= 243\left(\frac{\frac{(x^2+9)^{5/2}}{243}}{5} - \frac{\frac{(x^2+9)^{3/2}}{27}}{3}\right) + C$$

$$= \frac{(x^2+9)^{5/2}}{5} - 3(x^2+9)^{3/2} + C\,.$$

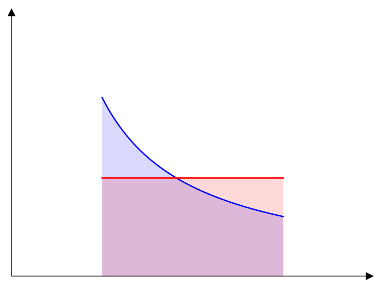
Remark: Some systems will simplify this further (see below) but the solution above suffices.

$$I = (x^2+9)^{3/2}\left[\frac{(x^2+9)}{5} - 3\right] + C = (x^2+9)^{3/2}\cdot\frac{(x^2-6)}{5} + C\,.$$

2.

(a) Average value is

$$\begin{aligned}\overline{y} &= \frac{1}{3-1}\int_1^3 \frac{1}{3x}\,dx = \left(\frac{1}{2}\cdot\frac{1}{3}\right)\int_1^3 \frac{1}{x}\,dx = \frac{1}{6}\left(\ln|x|\right)\Big|_1^3\\ &= \frac{1}{6}(\ln 3 - \ln 1) = \frac{1}{6}(\ln 3 - 0) = \frac{1}{6}\ln 3 = \ln \sqrt[6]{3}\,.\end{aligned}$$



The line is $y = \ln\sqrt[6]{3}$; the curve is $y = {}^1\!/_{3x}$.

(b) Net change in cost is

$$C(b) - C(a) = \int_a^b C'(x)\,dx\,.$$

We can use marginal cost as an approximation to $C'(x)$, so

$$\begin{aligned}C(3) - C(1) &= \int_1^3 (n-3)^2 - 1\,dn = \int_1^3 n^2 - 6n + 8\,dn\\ &= \left(\frac{n^3}{3} - 3n^2 + 8n\right)\Bigg|_1^3\\ &= \left[\left(\frac{27}{3} - 27 + 24\right) - \left(\frac{1}{3} - 3 + 8\right)\right]\\ &= \frac{2}{3}.\end{aligned}$$

The net change in cost would be roughly \$667.

(c) First we make sure there are no intersections to worry about:

$$\begin{aligned}4 - x &= \sqrt{1 - x^2}\\ 16 - 8x + x^2 &= 1 - x^2\\ 2x^2 - 8x + 15 &= 0\,.\end{aligned}$$

This is a quadratic equation. We can solve for x using the quadratic formula:

$$x = \frac{8 \pm \sqrt{8^2 - 4\times 2\times 15}}{2\times 2} \approx \frac{8 \pm \sqrt{64 - 120}}{4}\,.$$

A negative in the square root implies that the roots are complex (have imaginary parts) so in fact there is no intersection. We proceed to computing the area:

$$A=\int_0^1 (4-x)-\sqrt{1-x^2}\,dx=\int_0^1 (4-x)\,dx-\int_0^1 \sqrt{1-x^2}\,dx\,.$$

The first integral is straightforward:

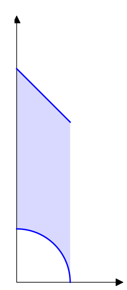
$$\int_0^1 4-x\,dx=\left(4x-\frac{x^2}{2}\right)\Big|_0^1=\left[\left(4\cdot 1-\frac{1^2}{2}\right)-\left(4\cdot 0-\frac{0^2}{2}\right)\right]=\frac{7}{2}\,.$$

The second integral is algebraically impossible for you at the moment, but if you recognize that $\sqrt{1-x^2}$ comes from a circle of radius 1 at the original, and the integral asks for the top-right *quarter* (not half!), then it's easy:

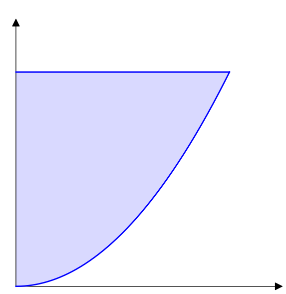
$$\int_0^1 \sqrt{1-x^2}\,dx=\frac{\pi\cdot 1^2}{4}=\frac{\pi}{4}\,.$$

So the area is

$$\frac{7}{2}-\frac{\pi}{4}\,.$$



(d) The base of the solid looks like this:



The cross sections perpendicular to the x-axis are squares with side length $s=1-x^2$. So the volume is

$$V=\int_0^1 B(x)\,dx=\int_0^1 s^2\,dx=\int_0^1 \left(1-x^2\right)^2\,dx=\int_0^1 1-2x^2+x^4\,dx$$

$$=\left(x-\frac{2x^3}{3}+\frac{x^5}{5}\right)\Big|_0^1=\left[\left(1-\frac{2\cdot 1^3}{3}+\frac{1^5}{5}\right)-\left(0-\frac{2\cdot 0^3}{3}+\frac{0^5}{5}\right)\right]=\frac{8}{15}\,.$$

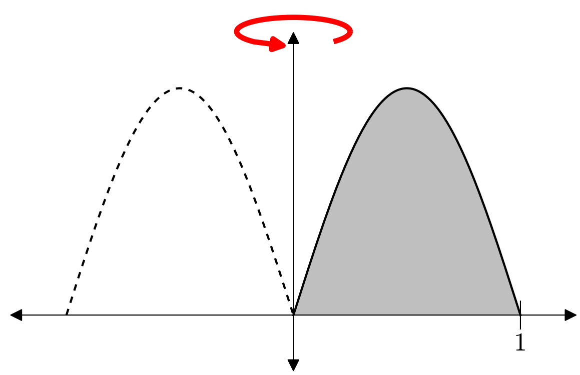
(e) If we rotate the area in part (c) about the x-axis, the resulting solid of revolution will have volume

$$V = V_{\text{outer}} - V_{\text{inner}} = \pi \int_0^1 (4-x)^2 \, dx - \pi \int_0^1 \left(\sqrt{1-x^2}\right)^2 \, dx \, .$$

(The first integral is the outer volume, the second integral is the inner volume of the hole.) This is easy enough to integrate:

$$\begin{aligned} V &= \pi \left[\int_0^1 16 - 8x + x^2 \, dx - \int_0^1 1 - x^2 \, dx \right] \\ &= \pi \left[\left(16x - 4x^2 + \frac{x^3}{3} \right) \Big|_0^1 - \left(x - \frac{x^3}{3} \right) \Big|_0^1 \right] \\ &= \pi \left[\left[\left(16 - 4 + \frac{1}{3} \right) - (0) \right] - \left[\left(1 - \frac{1}{3} \right) - (0) \right] \right] \\ &= \pi \left(12\frac{1}{3} - \frac{2}{3} \right) \\ &= \frac{35\pi}{3} \, . \end{aligned}$$

(f) The graph of $\sin(\pi x)$ intersects the axis at $x = 0$ and $x = 1$, so the shape is well-defined:



We are rotating about the y-axis, and we want to use shells, so we integrate with respect to x. (Shells integrates *perpendicular* to the axis of rotation.) That gives us

$$V = 2\pi \int_0^1 x \sin(\pi x) \, dx \, .$$

This looks like a product, so we use integration by parts:

$$\begin{aligned} u &= x & v' &= \sin(\pi x) \\ u' &= 1 & v &= -\frac{1}{\pi} \cdot \cos(\pi x) \end{aligned}$$

Hence

$$\begin{aligned}
V &= 2\pi\left[-\frac{x\cos(\pi x)}{\pi} - \int 1\times -\frac{\cos(\pi x)}{\pi}\,dx\right]\Bigg|_0^1 \\
&= 2\cancel{\pi}\left[-\frac{x\cos(\pi x)}{\cancel{\pi}} + \frac{1}{\cancel{\pi}}\int \cos(\pi x)\,dx\right]\Bigg|_0^1 \\
&= 2\left(-x\cos(\pi x) + \frac{1}{\pi}\sin(\pi x)\right)\Bigg|_0^1 \\
&= 2\left[\left(-1\cdot\cos\pi + \frac{1}{\pi}\cancelto{0}{\sin\pi}\right) - \left(-0\cos 0 + \frac{1}{\pi}\cancelto{0}{\sin 0}\right)\right] \\
&= 2\cdot(-1\times -1) \\
&= 2\,.
\end{aligned}$$

(g) If we rotate the area in part (c) about the y-axis, the resulting solid of revolution will have volume

$$V = 2\pi\int_0^1 x\left[(4-x) - \sqrt{1-x^2}\right]\,dx\,.$$

This is easy enough to integrate, requiring only one substitution:

$$\begin{aligned}
V &= 2\pi\int_0^1 4x - x^2 - x\sqrt{1-x^2}\,dx \\
&= 2\pi\left(2x^2 - \frac{x^3}{3} + \frac{1}{2}\cdot\frac{2}{3}\cdot\left(\sqrt{1-x^2}\right)^3\right)\Bigg|_0^1 \\
&= 2\pi\left[\left(2 - \frac{1}{3} + \frac{1}{3}\cdot 0\right) - \left(0 - 0 + \frac{1}{3}\cdot 1\right)\right] \\
&= 2\pi\left(\frac{5}{3} - \frac{1}{3}\right) \\
&= \frac{8\pi}{3}\,.
\end{aligned}$$