DEFINITIONS/BIG-TIME FACTS TO KNOW FOR TEST 1

CALCULUS II

Definitions

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line tangent to $f(x)$ at $x=a$

(geometric) a line that intersects the curve defined by f at $x=a$ *and* goes in the same direction as the curve defined by f

(algebraic) the line $y=m(x-a)+f(a)$ where $m=f'(a)$

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derivative of $f(x)$ at $x=a$

(geometric) the slope of the line tangent to f at a

(algebraic) $f'(x)=\dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0}\dfrac{\Delta x}{\Delta y}=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

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f is continuous on $[a,b]$

(geometric) f is *unbroken:* no gaps, jumps, or asymptotes

(precise) for any $c\in[a,b]$, we can find the limit of f at c by substitution; that is, $\lim\limits_{x\to c} f(x)=f(c)$

Example. Polynomials are always continuous. The rational function $f(x)={}^1/_x$ is continuous everywhere except $x=0$, where it has an asymptote.

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f is differentiable on (a,b)

(geometric) f is *smooth:* no corners, cusps, or breaks

(precise) for any $c\in(a,b)$, we can find the derivative of f at c

Example. Polynomials are always differentiable. The absolute-value function $f(x)=|x|$ is always continuous and differentiable everywhere except $x=0$.

Rieman Sum

(geometric) an approximation of the area under a curve using a finite number of rectangles

(algebraic) $\Sigma_{i=1}^{n} f\left(x_i^*\right)\Delta x$, where $\Delta x={}^{(b-a)}/_n$ and x_i^* is *any* point in the ith subinterval of width Δx of $[a,b]$

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choices of sample point for Riemann sums

right endpoint $x_i^*=a+i\Delta x$

left endpoint $x_i^*=a+(i-1)\Delta x$

midpoint $x_i^*=a+(i-{}^1/_2)\Delta x$

(definite) integral of $f(x)$ over $[a,b]$

(geometric) the *net* area between the curve of $f(x)$ and the x-axis, starting at $x=a$ and ending at $x=b$ ("net" means it can be negative or zero)

(algebraic) $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$, where $\Delta x = {}^{(b-a)}/_{n}$ and x_i^* is *any* point in the ith subinterval of width Δx of $[a,b]$, *as long as the limit exists*

Remark 1. We typically use right endpoints when computing the exact value of the definite integral.

Remark 2. Pay attention to what a problem asks. If it asks you to compute an integral using the definition, you *must* use the limit definition provided here. If, however, it asks you to compute or approximate an integral using geometry, you *must* make an argument using mere geometry, without recourse to limits or algebra. *You absolutely may not use any shortcuts that were discussed after we began discussing the Fundamental Theorem of Calculus.*

Example. The graph of $f(x)=\sqrt{a^2-x^2}$ is a semicircle from $[-a,a]$. Hence $\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = {}^{\pi a^2}/_{2}$ while $\int_0^a \sqrt{a^2-x^2}\,dx = {}^{\pi a^2}/_{4}$.

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f is **integrable over** $[a,b]$:

(geometric) we can find the area between the curve defined by f over $[a,b]$ and the x-axis

(algebraic) the limit $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ converges, where x_i^* and Δx are defined as above

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indeterminate form: any limit approaching the form ${}^{\pm\infty}/_{\pm\infty}$, ${}^{0}/_{0}$, $\infty-\infty$, $0\times\infty$, 0^0, 1^∞, ∞^0

Example. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ is an indeterminate form.

Big-time results or Tools

Fact (Interesting and important limits)**.**

(a) $\lim_{x\to\infty} 1/x = 0$

(b) If the limit of an expression approach zero/zero*, then there is more work to be done: either an algebraic massage, or L'Hôpital's Rule.*

(c) If the limit of an expression approaches nonzero/zero*, then*

- *(i) the one-sided limit is some sort of infinity, and*
- *(ii) the two-sided limit is either some sort of infinity, or doesn't exist (because the one-sided limits disagree).*

Theorem (L'Hôpital's Rule)**.** *If*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty} \qquad \text{or} \qquad \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad ,$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad .$$

Fact (Newton's Method)**.** *Let f be a differentiable function. If we know that $f(x_i) \approx 0$ — that is, x_i is close to a root of f — then we can often find a closer approximation to the root by*

(i) building a line tange to f at x_i, and

(ii) find the root (or x-intercept) of the tangent line.

We can express this as the following algorithm.

Algorithm (Newton's Method)**.** *Suppose f is differentiable and has a root on (a, b), which we want to approximate to d decimal places.*

(1) *Let $i = 0$ and choose a reasonable initial approximation x_0 ;*

(2) *let $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$;*

(3) *if the first d decimal places of x_i and x_{i+1} agree, then the root is approximately x_i ;*

(4) *otherwise, let $x_i = x_{i+1}$, add 1 to i, and return to step 2.*

Fact. *If $a, b \neq 0$ and the numerator and denominator of the first fraction are both polynomials of degree n, then*

$$\lim_{x\to\infty} \frac{ax^n + \cdots}{bx^n + \cdots} = \frac{a}{b} .$$

Proof. Notice that

$$\lim_{x\to\infty} \frac{ax^n + \cdots}{bx^n + \cdots} \to \frac{\infty}{\infty} ,$$

so L'Hôpital's Rule applies. We take the derivative of the numerator and denominator to obtain polynomials of degree $n-1$:

$$\lim_{x\to\infty} \frac{ax^n + \cdots}{bx^n + \cdots} \underset{\text{L'H}}{=} \lim_{x\to\infty} \frac{anx^{n-1} + \cdots}{bnx^{n-1} + \cdots} \to \frac{\infty}{\infty} .$$

Again L'Hôpital's Rule applies. We take the derivative of the numerator and denominator to obtain polynomials of degree $n-2$:

$$\lim_{x\to\infty} \frac{anx^{n-1} + \cdots}{bnx^{n-1} + \cdots} \underset{\text{L'H}}{=} \lim_{x\to\infty} \frac{an(n-1)x^{n-2} + \cdots}{bn(n-1)x^{n-2} + \cdots} \to \frac{\infty}{\infty} .$$

Again L'Hôpital's Rule applies. We can continue in this fashion until the degree of the numerator and denominator decreases to 0, at which point we have

$$\lim_{x\to\infty} \frac{an(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot x^0}{bn(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot x^0} .$$

What happened to the other terms in the numerator and denominator? Then had smaller degree, so they have already become zero. What little is left now reduces:

$$\lim_{x\to\infty} \frac{a\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots \cancel{3}\cdot \cancel{2}\cdot 1\cdot x^0}{b\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots \cancel{3}\cdot \cancel{2}\cdot 1\cdot x^0} = \lim_{x\to\infty} \frac{a}{b} .$$

Since a and b are nonzero constants,

$$\lim_{x\to\infty} \frac{ax^n + \cdots}{bx^n + \cdots} = \frac{a}{b} .$$

□

Example problems

This list is by no means exhaustive.

1. Compute the following limits. Use L'Hôpital's Rule *only* if necessary.

$$\lim_{x\to 0}\frac{x+\sin 8x}{x} \qquad \lim_{x\to 0}\frac{x+\sin 8x}{x^2} \qquad \lim_{x\to 0^+}\frac{\sqrt{x}}{\ln x} \qquad \lim_{x\to\infty}\frac{\sqrt{x}}{\ln x}$$

$$\lim_{x\to\infty}\left(x-\sqrt{x^2-9}\right) \qquad \lim_{x\to\infty}\left(x-\sqrt{x^2-9x}\right) \qquad \lim_{x\to 0^+} x^{1/x} \qquad \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$$

2. Approximate a root to each function, correct to the thousandths place. Start by using a graph to estimate an initial approximation.
   (a) $f(x)=x^3-2$
   (b) $g(x)=\tan x-\sin x$ (a *nonzero* root; $x=0$ is too easy)

3. (a) Use Newton's Method to approximate $\sqrt[3]{2}$ correct to the thousandths place.
   (b) Name two reasons Newton's Method can fail, *even when the function has a root.*

5. Find the indicated area **using only geometric methods**. *Explain* your computation. Drawing a picture would go a long way towards an explanation, but some words are probably necessary. Merely writing a formula is insufficient!

$$\int_a^b c\,dx \qquad \int_{-3}^3 x\,dx \qquad \int_2^5 x\,dx \qquad \int_0^6 (3-x)-\sqrt{36-x^2}\,dx$$

6. Approximate $\int_2^5 x\,dx$ **using three rectangles** and
   (a) left endpoints,
   (b) right endpoints, and
   (c) midpoints.
   Compare your answers to the corresponding integral in #4, and comment on the results (overestimate, underestimate, etc.).

7. Evaluate

$$\int_0^2 x^2-2x\,dx$$

   using the definition of the integral and right endpoints as the sample points. Is this an overestimate, an underestimate, or other? Why?

8. A particle travels at $v(t)=(x-2)^3$ meters/second from $t=0$ to $t=4$ seconds.
   (a) Use geometry to explain why the particle's displacement after four seconds is 0.
   (b) Divide the interval $[0,4]$ into four subintervals and use a Riemann sum with midpoints to approximate the particle's displacement. Explain why your approximation is so good in this case.
   (c) Would you have such a good approximation if you used right endpoints instead? Why or why not?
   (d) Use the definition of the integral and right endpoints to evaluate the displacement of the particle after 4 seconds. Notice that your answer should agree with part (a)!