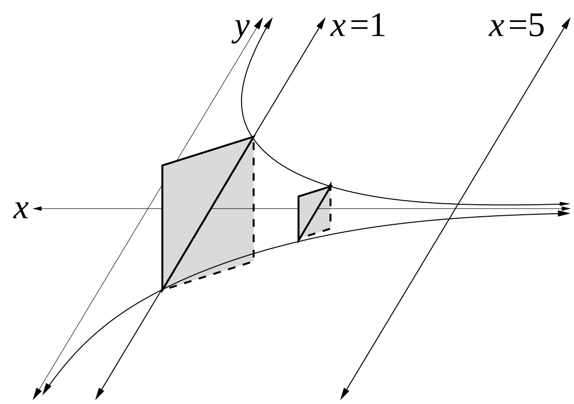
MAT 168 TEST 2 FORM A (APPLICATIONS OF INTEGRALS)

Directions: Solve each required problem **on a separate sheet of paper. Use pencil and show all work;** I deduct points for using pen or skipping important steps. Cell phones are not allowed; **you must shut off your cell phone.** Some problems are worth more than others. I encourage you to ask questions.

(1) Consider the following solid: its cross-sections are squares whose diagonals lie on the x-y plane; the solid is bounded above and below by the curves $10/x$ and $-10/x$; and the solid is bounded left and right by the lines $x = 1$ and $x = 5$. (See Figure 0.1.) Find the volume of the solid.

FIGURE 0.1.

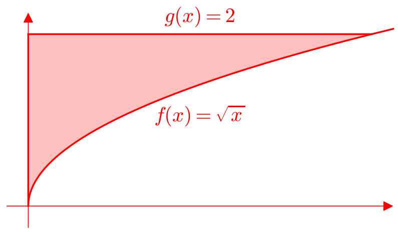


First off, isn't that an awesome graphic? I wish I remembered making it. Anyway, we can find the volume by slicing. Any of the solid's cross-sections perpendicular to the x-y plane is a square. The diagonal of the square has height $10/x - (-10/x)$, or $20/x$. So the volume is

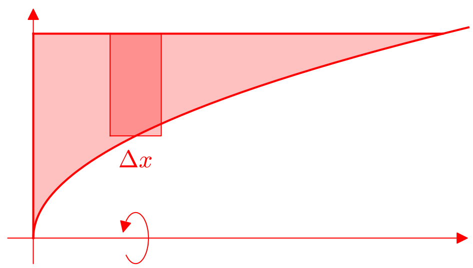
$$V = \int_1^5 \frac{20}{x}\,dx = 20 \ln|x| \Big|_1^5 = 20\left(\ln 5 - \cancel{\ln 1}^{\,0}\right) = 20 \ln 5 \,.$$

(2) The region defined by the y-axis and the functions $f(x) = \sqrt{x}$ and $g(x) = 2$ is rotated about the x-axis. Find the volume of the solid (a) using disks, and (b) using shells.

First let's look at a sketch of the region in question.



Disks: Finding volume by disks integrates along the same axis as the axis of rotation, with a rectangle perpendicular to the axis of rotation…



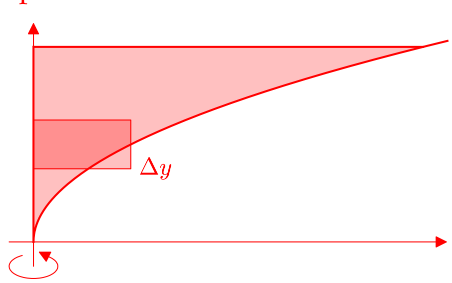
…so we need $\int \ldots dx$. Moreover, we need *washers*, because rotating that rectangle leaves a hole. The outer disk's volume will be $\pi \int_a^b 2^2 \, dx$, while the inner disk's radius will be $\pi \int_a^b \sqrt{x}^2 \, dx$. The interval begins at the x-value $a = 0$, and ends where f and g intersect:

$$\begin{aligned} \sqrt{x} &= 2 \\ x &= 4 \,. \end{aligned}$$

Hence $b = 4$ and

$$V = \pi \int_0^4 4 \, dx - \pi \int_0^4 x \, dx = \pi \left(4x - \frac{x^2}{2} \right) \Bigg|_0^4 = \pi \left[\left(16 - \frac{16}{2} \right) - (0 - 0) \right] = 8\pi \,.$$

Shells: Finding volume by shells integrates along the axis perpendicular to the axis of rotation, with a rectangle parallel to the axis of rotation…



…so we need $\int \dots dy$. The interval begins at the y-value $a = 0$, and ends at the y-value $b = 2$. Hence

$$V = 2\pi \int_0^2 y \cdot y^2 \, dy = 2\pi \int_0^2 y^3 \, dy = 2\pi \cdot \left. \frac{y^4}{4} \right|_0^2 = 2\pi \left(\frac{16}{4} - 0 \right) = 8\pi \,.$$

We found the same answer both ways, which is a good sign.

(3) Set up *but do not solve* an integral to find the circumference of a circle. (*Hint:* Another way of looking at this is that I'm asking you to find the *arclength* of the circle.)

The equation of a circle whose center is $(0,0)$ and whose radius is r is $x^2 + y^2 = r^2$. If we find the arclength of the top half of the circle, then we can find the circumference simply by multiplying by 2. The top half of the circle is described by $y = \sqrt{r^2 - x^2}$, and the giving us the arclength

$$s = \int_{-r}^{r} \sqrt{1 + \left(\frac{-2x}{2\sqrt{r^2 - x^2}} \right)^2} \, dx = \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, dx \,.$$

The problem does not ask you to compute the integral, but you can:

$$\begin{aligned} s &= \int_{-r}^{r} \sqrt{\frac{(r^2 - x^2) + x^2}{r^2 - x^2}} \, dx \\ &= \int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}} \, dx \\ &= \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}} \\ &= \int_{-r}^{r} \frac{1}{\sqrt{1 - \left(\frac{x}{r}\right)^2}} \, dx \\ &= r \arcsin \left. \left(\frac{x}{r} \right) \right|_{-r}^{r} \\ &= r \left(\arcsin 1 - \arcsin(-1) \right) \\ &= r \left(0 - (-\pi) \right) \\ &= \pi r \,. \end{aligned}$$

The circumference of a circle is thus $2 \times \pi r$.

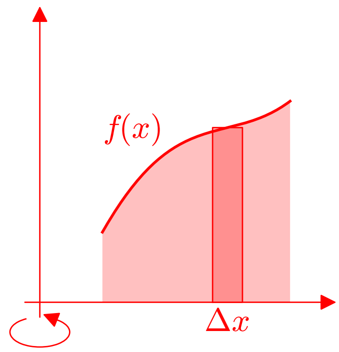
(4) Find the center of mass of a thin plate covering the region bounded below by the parabola $y = x^2$ and above by the line $y = x$, if the plate's density at any point (x, y) is $\delta(x) = 12x$. (*Note:* If you cannot remember the relevant formulas, you can "buy" them from me at a small penalty.)

Omit.

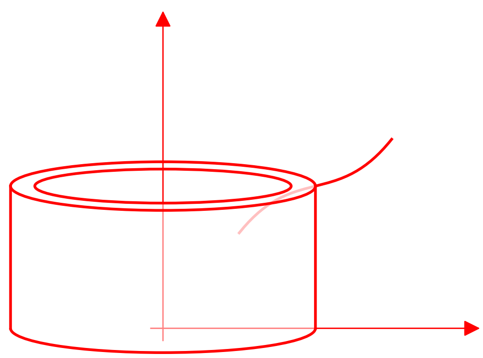
(5) Explain why the formula to find the volume of a surface of revolution by shells is $V = 2\pi \int_a^b x f(x) \, dx$. Your explanation should begin with an approximation by finite sums, using a diagram to illustrate the approximation.

I added a lot more than I expect from you on the test, so **the absolute minimum is in bold.**

We want to find the volume of the solid formed by rotating $f(x)$ about the y-axis. (We know it's the y-axis because the integral is with respect to x and a volume by shells is integrated along the axis perpendicular to the axis of rotation.)



When we rotate the indicated rectangle about the y-axis, we obtain something akin to a round shell.



Slice the shell and unroll to obtain a rectangle, so the shell's volume is roughly equivalent to that of a rectangular prism. **The rectangle's width is the shell's circumference and the rectangle's length is the shell's height.** The shell has radius x, so its circumference is $2\pi x$. The shell's height is $f(x)$. **The shell's volume is thus**

$$V_i = \underbrace{2\pi x_i}_{\text{width}} \times \underbrace{f(x_i)}_{\text{length}} \times \underbrace{\Delta x}_{\text{depth}} .$$

That makes the solid's volume approximately

$$V \approx \sum_{i=1}^{n} 2\pi x_i f(x_i) \Delta x .$$

To avoid error, take the limit as $n \to \infty$, and we have an integral formula:

$$V = \lim_{n\to\infty} \sum_{i=1}^{\infty} 2\pi x_i f(x_i) \Delta x = \int_a^b 2\pi x f(x)\, dx = 2\pi \int_a^b x f(x)\, dx .$$