## TEST 3 FORM A

MAT 168

Directions: Try to earn 100 points from the problems, writing in the blue examination book in pencil, and showing all work. Put away any cell phones; the mere appearance will give a zero.

This test went through 2 or 3 revisions before I printed it out, and while this final version is much, much better than the first drafts, I seem to have introduced one dumb error in \#1, and overlooked another in \#2. That's not a great way to start the test, and I apologize for that. I really did try to work out all the wrinkles.

1. ( $\mathbf{1 5} \mathbf{p t s}$ ) Find the area of the region bounded by the curves $f(x)=x$ and $g(x)=x^{2}-1$.

First we need to find the points of intersection, for which we set the curves equal:

$$
\begin{aligned}
& x=x^{2}-1 \\
& 0=x^{2}-x-1 \\
& x=\frac{1 \pm \sqrt{1+4}}{2}=\frac{1 \pm \sqrt{5}}{2} .
\end{aligned}
$$

To find the "top" function, we evaluate $f(0)=0$ and $g(0)=-1$, so $f$ is on top. The area is thus

$$
\begin{aligned}
\int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} x-\left(x^{2}-1\right) d x & =\left.\left(-\frac{x^{3}}{3}+\frac{x^{2}}{2}+x\right)\right|_{\frac{1-\sqrt{5}}{2}} ^{\frac{1+\sqrt{5}}{2}} \\
& =\left(-\frac{(1+\sqrt{5})^{3}}{24}+\frac{(1+\sqrt{5})^{2}}{4}+\frac{1+\sqrt{5}}{2}\right) \\
& -\left(-\frac{(1-\sqrt{5})^{3}}{24}+\frac{(1-\sqrt{5})^{2}}{4}+\frac{1-\sqrt{5}}{2}\right)
\end{aligned}
$$

This answer would suffice.
Note: I meant to choose $g(x)=x^{2}-2$, which would have made a much nicer answer. To wit:

$$
x=x^{2}-2 \quad \Longrightarrow \quad 0=x^{2}-x-2 \quad \Longrightarrow \quad 0=(x-2)(x+1)
$$

and then $x=-1,2$. This typo made the problem harder than I meant it to be, and I will take that into account when grading it.
2. (20 pts) The base of a solid lies between the graphs of $y=2 \sqrt{1-\left(x^{2} / 9\right)}, y=0$, and $x=0$. Cross sections perpendicular to the $x$-axis are circles. Use slicing to find the solid's volume.

If you sketch the region, you have something like this:


Unfortunately, this specification is ambiguous: there is a clear region between $y=2 \sqrt{1-\left(x^{2} / 9\right)}$ and $y=0$ (the $x$-axis), but when you throw in $x=0$ (the $y$-axis) it isn't clear whether you're supposed to choose the left half, right half, both, or neither.

I had actually meant the right half of the solid, like so:


Once I again I have to apologize if this confused anyone; I will certainly accept it if you worked with the entire ellipse. In any case, the slices are circles; the radius is

$$
r=\frac{2 \sqrt{1-\frac{x^{2}}{9}}}{2}=\sqrt{1-\frac{x^{2}}{9}}
$$

because the radius has to be half the solid. That means

$$
V=\int_{0}^{3} \pi\left(\sqrt{1-\frac{x^{2}}{9}}\right)^{2} d x=\pi \int_{0}^{3} 1-\frac{x^{2}}{9} d x=\left.\pi\left(x-\frac{x^{3}}{27}\right)\right|_{0} ^{3}=\pi\left[\left(3-\frac{27}{27}\right)-0\right]=2 \pi .
$$

3. A solid is formed by rotating the region bounded by $y=x^{4}$ and $y=\sqrt{x}$ about the $x$-axis.

Before we proceed, let's diagram the region that is being rotated.


We need to find the points of intersection, though you can probably guess them in this case:

$$
\begin{aligned}
x^{4} & =\sqrt{x} \\
x^{8} & =x \\
x^{8}-x & =0 \\
x\left(x^{7}-1\right) & =0 \\
x & =0,1 .
\end{aligned}
$$

(a) (20 pts) Use the method of discs and washers to find the volume of the solid.

The axis of integration is the axis of rotation, so we integrate with respect to $x$. You can also probably guess which curve is on top, though this is easily determined by substituting $x=1 / 2$ into each:

$$
\left(\frac{1}{2}\right)^{4}=\frac{1}{16}<\frac{1}{\sqrt{2}}=\sqrt{\frac{1}{2}} \quad \Longrightarrow \quad \sqrt{x} \text { is on top. }
$$

We have

$$
\begin{aligned}
V_{\text {top }}-V_{\text {bot }} & =\pi \int_{0}^{1}(\sqrt{x})^{2} d x-\pi \int_{0}^{1}\left(x^{4}\right)^{2} d x \\
& =\pi\left(\int_{0}^{1} x d x-\int_{0}^{1} x^{8} d x\right) \\
& =\pi\left(\left.\frac{x^{2}}{2}\right|_{0} ^{1}-\left.\frac{x^{9}}{9}\right|_{0} ^{1}\right) \\
& =\pi\left[\left(\frac{1}{2}-0\right)-\left(\frac{1}{9}-0\right)\right] \\
& =\frac{7 \pi}{18}
\end{aligned}
$$

(b) ( $\mathbf{1 5} \mathbf{~ p t s ) ~ U s e ~ t h e ~ m e t h o d ~ o f ~ s h e l l s ~ t o ~ f i n d ~ t h e ~ v o l u m e ~ o f ~ t h e ~ s o l i d . ~}$

The axis of integration is perpendicular to the axis of rotation, so first we solve for $x$ in terms of $y$, obtaining

$$
x=y^{2} \quad \text { and } \quad x=\sqrt[4]{y} .
$$

The limits of integration are still the same, and the larger function is $\sqrt[4]{y}$. We have

$$
\begin{aligned}
V & =2 \pi \int_{0}^{1} y\left(\sqrt[4]{y}-y^{2}\right) d y \\
& =2 \pi \int_{0}^{1} y^{\frac{5}{4}}-y^{3} d y \\
& =\left.2 \pi\left(\frac{y^{9 / 4}}{9 / 4}-\frac{y^{4}}{4}\right)\right|_{0} ^{1} \\
& =2 \pi\left[\left(\frac{1}{9 / 4}-\frac{1}{4}\right)-0\right] \\
& =2 \pi\left(\frac{4}{9}-\frac{1}{4}\right) \\
& =2 \pi \times \frac{7}{36} \\
& =\frac{7 \pi}{18}
\end{aligned}
$$

4. ( $7 \mathbf{p t s}$ ) Set up, but do not simplify, an integral to find the arclength of $f(x)=\cos x$ on $[0, \pi / 2]$.

We need $f^{\prime}(x)=-\sin x$. Then

$$
s=\int_{0}^{\pi / 2} \sqrt{1+\sin ^{2} x} d x
$$

We did not need to simplify it, so we stop there.
5. (8 pts) Set up, but do not simplify, an integral to compute the surface area of $f(x)=\cos x$ on $[0, \pi / 2]$.

We need $f^{\prime}(x)=-\sin x$. Then

$$
s=2 \pi \int_{0}^{\pi / 2} \cos x \sqrt{1+\sin ^{2} x} d x
$$

We did not need to simplify it, so we stop there.
6. ( $\mathbf{1 0} \mathbf{~ p t s}$ ) A carpet which is 5 meters long is completely rolled up. When $x$ meters have been unrolled, the force required to unroll it further is

$$
F(x)=\frac{450}{(x+1)^{2}} \mathrm{~N}
$$

How much work is done unrolling the entire carpet? Your answer must include the correct units.

Work is the integral of force, so

$$
W=\int_{0}^{5} \frac{450}{(x+1)^{2}} d x
$$

Let $u=x+1$; then $d x=d u$. When $x=0, u=1$, and when $x=5, u=6$. So the integral is equivalent to

$$
W=450 \int_{1}^{6} \frac{1}{u^{2}} d u=450 \times\left.\left(-\frac{1}{u}\right)\right|_{1} ^{6}=450\left[\left(-\frac{1}{6}-(-1)\right)\right]=450 \times \frac{5}{6}=75 \times 5=375 \mathrm{~J} .
$$

7. (20 pts) A tank in the shape of a trapezoidal prism is completely filled with 3 m of soft serve ice cream (see sketch for dimensions). Find the work required to empty the tank by pumping the soft serve over the top of the tank. The density of soft serve is 727 $\mathrm{kg} / \mathrm{m}^{3}$.

We model this problem with "slabs" of soft serve (the dashed line was supposed to hint at this). Consider the slab whose height from the tank bottom is $h_{i}$. It has to travel $3-h_{i}$ meters. It has the shape of a rectangular prism whose depth is $\Delta h$. The slab's length is 5 m .


The width is a little harder: at its minimum, it is 1 m , and it rises over 3 m to a width of 1.5 m That gives us a slope of $5 / 3$, so the width is $1+(.5 / 3) h_{i}=1+(1 / 6) h_{i}$. (Another way to figure
this out is via similar triangles, as with the cone of hot chocolate in the homework problem.) Hence the volume of the $i$ th slab is

$$
V_{i}=5\left(1+\frac{h_{i}}{6}\right) \Delta h
$$

The mass of the $i$ th slab is

$$
m_{i}=\rho V_{i}=727 \times 5\left(1+\frac{h_{i}}{6}\right) \Delta h=3635\left(1+\frac{h_{i}}{6}\right) \Delta h .
$$

The force required to overcome the force of gravity on the $i$ th slab is

$$
F_{i}=m_{i} \times g=3635\left(1+\frac{h_{i}}{6}\right) \Delta h \times 9.8=35623\left(1+\frac{h_{i}}{6}\right) \Delta h .
$$

The work performed in lifting the $i$ th slab is

$$
W_{i}=F_{i} \times d_{i}=35623\left(1+\frac{h_{i}}{6}\right) \Delta h \times\left(3-h_{i}\right) .
$$

The total work is

$$
W \approx \sum_{i=1}^{n} 35623\left(1+\frac{h_{i}}{6}\right)\left(3-h_{i}\right) \Delta h .
$$

To obtain the exact value, compute the limit as $n \rightarrow \infty$, which turns it into an integral.

$$
\begin{aligned}
W & =35623 \int_{0}^{3}\left(1+\frac{h}{6}\right)(3-h) d h \\
& =35623 \int_{0}^{3} 3-\frac{h}{2}-\frac{h^{2}}{6} d h \\
& =\left.35623\left(3 h-\frac{h^{2}}{4}-\frac{h^{3}}{18}\right)\right|_{0} ^{3} \\
& =35623\left[\left(9-\frac{9}{4}-\frac{27}{18}\right)-0\right] \\
& =35623 \times \frac{21}{4} \\
& =187020.75 \mathrm{~J}
\end{aligned}
$$

8. ( $\mathbf{1 5} \mathbf{~ p t s ) ~ C h o o s e ~ o n e ~ o f ~ t h e ~ a p p l i c a t i o n s ~ o f ~ i n t e g r a t i o n ~ l i s t e d ~ b e l o w . ~ E x p l a i n ~ h o w ~ t o ~ d e r i v e ~ t h e ~}$ formula for the application. The explanation need not cover every mathematical detail, but it should hit the four main points of turning a model into an integral.
(a) volume by discs
(b) arclength
(c) average value
(d) the mass of a rod

Omitted.

