TEST 1 FORM A

MAT 168

Directions: Solve as many problems as well as you can in the blue examination book, writing in pencil and showing all work. Put away any cell phones; the mere appearance will give a zero.

1. What does it mean for a function $f(x)$ to be **continuous** on an interval $[a,b]$? Give both a geometric and an algebraic definition.

   geometric: the graph of f is unbroken on $[a,b]$; put another way, it has no holes, skips, or asymptotes

   algebraic: we can find the limit of f at any point $c \in [a,b]$ by substitution; put another way, for any $c \in [a,b]$,

$$\lim_{x\to c} f(x) = f(c)$$

1. *(second version of the test)* What does it mean for a function $f(x)$ to be **integrable** on an interval $[a,b]$? Give both a geometric and an algebraic definition.

   geometric: we can compute the area between $f(x)$ and the x-axis on $[a,b]$

   algebraic: the limit of the Riemann sum of f over $[a,b]$ exists; put another way,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \quad \text{exists} .$$

2. Compute the following limits, if they exist. Use L'Hôpital's Rule *only* if necessary.

   (a) $\displaystyle\lim_{x\to -1} \frac{x^2+2x+1}{x^2-1}$

   As $x \to -1$, the numerator and denominator both approach 0. Hence L'Hôpital's Rule applies, and we can compute

$$\lim_{x\to -1} \frac{x^2+2x+1}{x^2-1} \overset{\text{L'H}}{=} \lim_{x\to -1} \frac{2x+2^{\nearrow 0}}{2x_{\searrow -2}} = 0 .$$

   (b) $\displaystyle\lim_{x\to\infty} x^2 e^{-2x}$

   As $x \to \infty$, this limit approaches the indeterminate form $\infty \cdot 0$. We need to rewrite it as a fraction; then we can apply L'Hôpital's Rule on ∞/∞:

$$\lim_{x\to\infty} \frac{x^2{}^{\nearrow\infty}}{e^{2x}{}_{\searrow\infty}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{2x^{\nearrow\infty}}{2e^{2x}{}_{\searrow\infty}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{2^{\nearrow 2}}{4e^{2x}{}_{\searrow\infty}} = 0 .$$

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(c) $\displaystyle\lim_{x\to\frac{\pi}{2}^-}\frac{\tan x}{\cos x}$

As $x\to\infty$, this limit approaches the form $\infty/0$. Non-zero over zero is always some sort of infinity. Since the denominator is always positive, the limit approaches ∞.

(d) $\displaystyle\lim_{x\to\infty}\left(1+\frac{5}{x}\right)^{2x}$

As $x\to\infty$, $5/x\to 0$, so this limit approaches the indeterminate form 1^∞. To resolve it we set $y=(1+5/x)^{2x}$, take the natural logarithm of both sides, and consider the limit of this new expressions:

$$\lim_{x\to\infty}(\ln y)=\lim_{x\to\infty}\left[\ln\left(1+\frac{5}{x}\right)^{2x}\right]=\lim_{x\to\infty}\left[2x\ln\left(1+\frac{5}{x}\right)\right].$$

This approaches the indeterminate form $\infty\cdot 0$. We need to rewrite it as a fraction; then we can apply L'Hôpital's Rule on $0/0$:

$$\lim_{x\to\infty}(\ln y)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{5}{x}\right)^{\nearrow 0}}{\frac{1}{2x}_{\searrow 0}}\overset{\text{L'H}}{=}\lim_{x\to\infty}\frac{\frac{1}{1+\frac{5}{x}}\cdot 5\cdot\left(-\frac{1}{x^2}\right)}{\frac{1}{2}\cdot\left(-\frac{1}{x^2}\right)}=\lim_{x\to\infty}\left(\frac{5}{1+\frac{5}{x}}\right)\cdot\frac{2}{1}=\lim_{x\to\infty}\frac{10^{\nearrow 10}}{1+\frac{5}{x}_{\searrow 1}}=10.$$

Remember that this is the limit of $\ln y$, not of y itself. Fortunately, the natural logarithm is continuous, so we can rewrite this as

$$\lim_{x\to\infty}y=\lim_{x\to\infty}e^{\ln y}=e^{\lim_{x\to\infty}(\ln y)}=e^{10}.$$

(e) The figure at right shows two functions that intersect at $(0,0)$. Estimate

$$\lim_{x\to 0}\frac{f(x)}{g(x)}.$$

Since the functions intersect at the origin, we know that

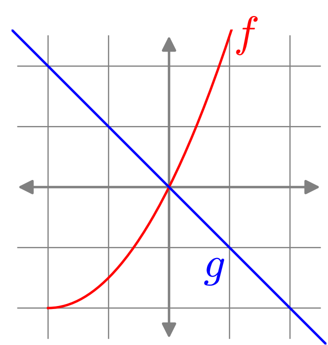
$$\lim_{x\to 0}\frac{f(x)}{g(x)}\begin{matrix}\to\\\to\end{matrix}\frac{0}{0},$$

so L'Hôpital's Rule applies directly:

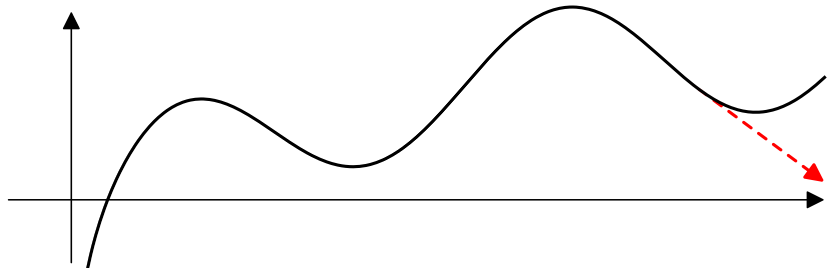
$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}.$$

We know neither f' nor g', but we can find $f'(0)$ and $g'(0)$ because they are the slopes of the lines tangent to f and g at $x=0$. Draw the tangent lines for those curves at $(0,0)$ and it looks as if the tangent line for f has slope 2 while the tangent line for g has slope -1 (I'd be flexible in how close you come to these values). It seems that

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\frac{2}{-1}=-2.$$



3. Suppose we want to approximate the root of $y = \ln x + \sin x$ using Newton's method.
   (a) Does $x = 10$ seem like a good starting position? Explain your answer.
   It really helps to look at the graph around $x = 10$. (It also helps to be in radian mode, so that you have the correct graph.)

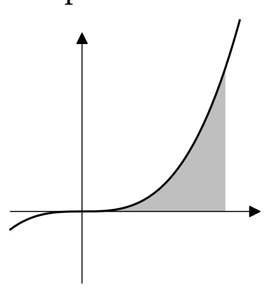


   It also helps to remember that Newton's method works by finding the root of a line tangent to f at the starting point. In this sketch, the tangent line is in red, and we see quite plainly that it carries one *away* from the root on the left, and that moreover the curve has lots of dips around $x = 10$ that could lead to infinite loops.
   (b) Starting with $x = 1$, find the next four approximations.
   Applying the formula $x_{i+1} = x_i - f(x_i)/f'(x_i)$ should give you the values 0.4537, 0.5671, 0.5786, 0.5787.
   (c) Is the last approximation correct to the nearest thousandth place? Why or why not?
   We have good reason to believe so: the thousandths place has begun to repeat.

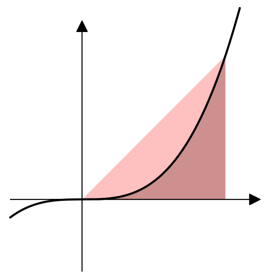
4. We want to find or approximate

$$A = \int_0^1 x^3 \, dx \, .$$

   (a) Sketch a graph of the region. Do we expect the area to be positive or negative?



   We expect the area to be positive, because it lies completely above the x-axis.
   (b) Use geometry to approximate the integral. Is your approximation an over- or underestimate?
   One might use a triangle: $A = 1/2 \cdot 1 \cdot 1 = 1/2$. This is an overestimate, because the region lies entirely within the triangle:



(c) Use four rectangles and left endpoints to approximate A to the thousandths place.
With left endpoints, $A \approx [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\,\Delta x$ where $\Delta x = {}^{(1-0)}/_{4} = {}^{1}/_{4}$ and $x_i = a + (i-1)\,\Delta x = {}^{(i-1)}/_{4}$. So

$$A \approx \left[f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right] \cdot \frac{1}{4} = \left(0 + \frac{1}{64} + \frac{1}{8} + \frac{27}{64}\right) \cdot \frac{1}{4} = \frac{36}{256} = \frac{9}{16}\,.$$

(d) Use the **definition of the integral** to find the exact value of A.
We have

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = {}^{(b-a)}/_{n} = {}^{1}/_{n}$ and $x_i = a + i\Delta x = {}^{i}/_{n}$. Hence

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^3} \cdot \frac{1}{n} \\
&= \lim_{n\to\infty} \left[\frac{1}{n^4} \sum_{i=1}^{n} i^3\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{n^4} \cdot \frac{n^2(n+1)^2}{4}\right] \\
&= \lim_{n\to\infty} \frac{n^4 + \text{stuff we don't care about}}{4n^4} \\
&= \frac{1}{4}\,.
\end{aligned}$$

5. The table at right gives several values of a function $f(x)$ on the interval $[2,5]$.

x	2	2.5	3	3.5	4	4.5	5
$f(x)$	10	5	2	1	2	4	3

(a) Use this data with a right-hand Riemann Sum and 3 sample points to approximate $\int_2^5 f(x)\,dx$.
It's important to divide the interval into 3 equal parts; that gives us $\Delta x = {}^{(5-2)}/_{3} = 1$. With right-hand endpoints, then, the sample points would be $x = 3, 4, 5$ so

$$\int_2^5 f(x)\ dx \approx \sum_{i=1}^{3} f(x_i)\,\Delta x = f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1 = 7\,.$$

(b) Can the table give us a more accurate approximation? Why or why not?
Yes. The table has a lot of data we didn't use, so we could obtain a more accurate approximation with 6 sample points, where $\Delta x = {}^{(5-2)}/_{6} = {}^{1}/_{2}$.

Useful formulas

Left endpoints: $x_i^* = a + (i-1)\,\Delta x$
Right endpoints: $x_i^* = a + i\Delta x$
Midpoints: $x_i^* = a + \left(i - \frac{1}{2}\right)\Delta x$
Sum shortcuts:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{i\,(i+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n\,(n+1)\,(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2\,(n+1)^2}{4}$$