MAT 168 FINAL EXAM

Directions: Solve each problem. **Use pencil and show all work;** I deduct points for using pen or skipping important steps. Cell phones are not allowed, not even as calculators; **you must shut off your cell phone.** If you find a problem challenging, save it for later and find something you can do more easily, to avoid running out of time. Some problems are worth more than others. I encourage you to ask questions.

Name: ____

1.1. What does it mean for a function $f(x)$ to be differentiable on an interval $[a,b]$? Give both a geometric and an algebraic definition.

geometric: the graph of f is *smooth* on (a,b); put another way, it has no breaks, corners, or kinks

algebraic: we can find the derivative of f at any point in (a,b); put another way, one of the following limits exists at every $c \in (a,b)$ (and since they are equivalent, all of them exist):

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

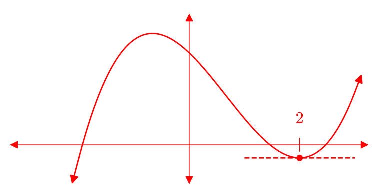
1.2. State the difference between a "proper" and an "improper" integral.

A proper integral has finite limits of integration and no asymptotes. An improper integral has an infinite limit of integration or an asymptote.

1.3. Suppose we want to approximate a positive root of $x^3 - 2x^2 - 4x + 7$ using Newton's Method.

(a) Why would $x_0 = 2$ be a catastrophic place to start? (There are at least two reasons, but I'll take just one.)

Let's look at the graph.



(1) Consider the derivative, $3x^2 - 4x - 4$; at $x = 2$ it is $3 \times 2^2 - 4 \times 2 - 4 = 0$. In other words, the tangent line is horizontal, and has no root, so it will not approximate a root of the curve. Another way of saying this is that you end up with division by zero in the formula $x_i - {}^{f(x_i)}/_{f'(x_i)}$.

(2) The point $(2, f(2))$ appears in a "bowl", and bowls can lead Newton's method into infinite loops. (To see why, shift the graph up so that the bowl is above the x-axis, and experiment with the behavior of the tangent lines.)

(a) Find the first four approximations when we start with $x_0 = 1$, instead.

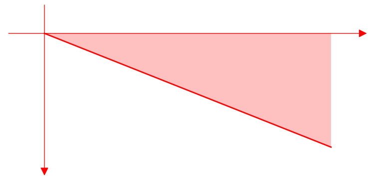
$x_1 = 1.400$, $x_2 = 1.4602$, $x_3 = 1.4626$, $x_4 = 1.4626$

(b) Is the last approximation correct to the nearest thousandth place? Why or why not?

We believe the approximation is correct to the nearest thousandth because the decimals approximation has begun to repeat there — in fact, it repeats even in the ten and hundred thousandth place!

1.4. Use **geometry** to find $\int_0^b cx\,dx$, where b and c are both positive. Some words should explain the reasoning.

The graph of the function is a line whose slope is c and whose y-intercept is 0. Over the interval $[0, b]$, the geometric figure is a triangle, either above or below the x-axis. For instance:



The triangle's base has length b. The triangle's height has length bc. From the formula for the area of a triangle we conclude that

$$\int_0^b cx\,dx = \frac{1}{2}b^2c\,.$$

1.5. Compute the following limits. Use L'Hospital's Rule *only* when necessary.

(a) $\lim\limits_{x\to 1} \dfrac{3\,(x-1)}{\ln x}$

$$\lim_{x\to 1} \frac{3\,(x-1)^{\nearrow 0}}{\ln x_{\searrow 0}} \overset{\text{L'H}}{=} \lim_{x\to 1} \frac{3}{1/x} = 3$$

(b) $\lim\limits_{x\to 1^-} \dfrac{x+1}{x-1}$

$$\lim_{x\to 1^-} \frac{x+1^{\nearrow 2}}{x-1_{\searrow 0^-}} = -\infty$$

(c) $\lim\limits_{x\to 0^+} x\cot 2x$

$$\lim_{x\to 0^+} x\cot 2x^{\nearrow 0\cdot\infty} = \lim_{x\to 0^+} \frac{x^{\nearrow 0}}{\tan 2x_{\searrow 0}} \overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{1}{2\sec^2 2x} = \frac{1}{2}$$

(d) $\lim\limits_{x\to 0^+} (1+x)^{\frac{3}{x}}$

Let $y = (1+x)^{3/x}$ and consider $\ln y = (3/x)\ln(1+x)$. We have

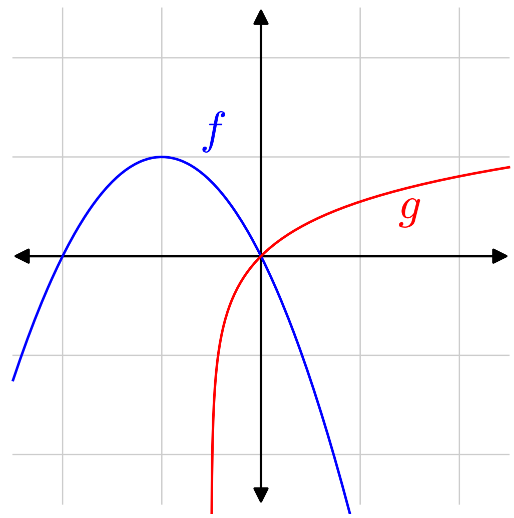
$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{3\ln(1+x)^{\nearrow 0}}{x_{\searrow 0}} \overset{\text{L'H}}{=\!=} \lim_{x\to 0^+} \frac{3/1+x}{1} = 3\,.$$

However, we don't want the limit of $\ln y$; we want the limit of y. We can find that:

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+}\ln y} = e^3\,.$$

(e) The figure at right shows two functions that intersect at $(0,0)$. Estimate

$$\lim_{x\to 0} \frac{f(x)}{g(x)}.$$



Precisely because the functions intersect at $(0,0)$ we know that we can use L'Hôpital's Rule. To find the values of the derivative at a point, sketch in a tangent line; you should have $f'(0) = -2$ and $g'(0) = 1$. Hence

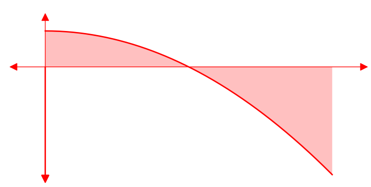
$$\lim_{x\to 0} \frac{f(x)^{\nearrow 0}}{g(x)_{\searrow 0}} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{f'(0)}{g'(0)} = \frac{-2}{1} = -2\,.$$

1.6. We want to find or approximate

$$A = \int_0^2 1 - x^2 \, dx$$

without using integration shortcuts. Certain useful formulas for this task appear on the following page.

(a) Sketch the region in question. Do you think the integral be positive or negative? Why?



The integral should be negative, as it looks as if there is more area below the axis than above.

(b) Use high school geometry to approximate the integral. Explain your work with words. *To be clear: I do not want anything sophisticated here. I do not expect you to find the exact area in this step, or even to find a particularly accurate approximation. Your grade depends on how intelligently you use ideas of high school geometry to approximate the area. As long as it makes sense, you earn full credit.*

I would approximate this with a quarter circle of radius 1 and a triangle of length 1 and height 3. The triangle, however, will have negative area because it lies below the x-axis.

$$A \approx \frac{1}{4} \times \pi \times 1^2 + \frac{1}{2} \times 1 \times 3 = \frac{\pi}{4} + \frac{3}{2}\,.$$

(c) Use four rectangles and left endpoints to approximate A.

$$A \approx \left[\left(1-0^2\right)+\left(1-({}^1\!/\!{}_2)^2\right)+\left(1-1^2\right)+\left(1-({}^3\!/\!{}_2)^2\right)\right] \times \frac{1}{2}$$
$$= [1+({}^3\!/\!{}_4)+0+(-{}^5\!/\!{}_4)] \times \frac{1}{2}$$
$$= {}^1\!/\!{}_8$$

(d) Use four rectangles and midpoints to approximate A.

$$A \approx \left[\left(1-({}^1\!/\!{}_4)^2\right)+\left(1-({}^3\!/\!{}_4)^2\right)+\left(1-({}^5\!/\!{}_4)^2\right)+\left(1-({}^7\!/\!{}_4)^2\right)\right] \times \frac{1}{2}$$
$$= [({}^{15}\!/\!{}_{16})+({}^7\!/\!{}_{16})+({}^{-9}\!/\!{}_{16})+({}^{-33}\!/\!{}_{16})] \times \frac{1}{2}$$
$$= -{}^5\!/\!{}_8$$

(e) Use the definition of the integral to find the exact value of A.
Remark: Use the **definition of the integral**, not an **integration shortcut**. Using an integration shortcut will earn you 0 points *even if you find the correct value.*

$$\begin{aligned}
\int_0^2 \left(1-x^2\right) dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a+i\Delta x\right) \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(i \cdot \frac{2}{n}\right) \cdot \frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[1-\left(\frac{2i}{n}\right)^2\right] \cdot \frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1-\frac{4i^2}{n^2}\right) \cdot \frac{2}{n} \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{2}{n} - \sum_{i=1}^{n} \frac{8i^2}{n^3}\right) \\
&= \lim_{n\to\infty} \left(\frac{2}{n}\sum_{i=1}^{n} 1 - \frac{8}{n^3}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \left(\frac{2}{n} \times n - \frac{8}{n^3} \times \frac{n\left(n+1\right)\left(2n+1\right)}{6}\right) \\
&= \lim_{n\to\infty} \left(2 - \frac{16n^3 + \text{stuff}}{6n^3}\right) \\
&= 2 - \frac{16}{6} \\
&= -\frac{2}{3}\,.
\end{aligned}$$

(f) Why do we expect (d) to be more accurate than (c), and (e) to be exact?
We expect (d) to be more accurate because the midpoint method is more accurate. (If pressed, I would say that most of the error cancels out in each rectangle, whereas with left and right endpoints, in general error accumulates across the rectangles.)
We expect (e) to be exact because increasing the number of rectangles reduces the error, and taking the limit eliminates the error completely.

Left endpoints: $x_i^* = a + (i-1)\,\Delta x$
Right endpoints: $x_i^* = a + i\Delta x$
Midpoints: $x_i^* = a + \left(i - \frac{1}{2}\right)\Delta x$
Sum shortcuts:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n\,(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n\,(n+1)\,(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2\,(n+1)^2}{4}$$

2.4. Compute **eight** of the antiderivatives indicated. Some require integration by u-substitution; others, integration by parts; still others, trigonometric techniques, including trigonometric substitution. Most are proper, but a few were brought up without manners and are thus improper. As in real life, it's not always obvious who's proper and who ain't.

(a) $\int \cos^4 x \sin x \, dx$
Let $u = \cos x$. Then ${}^{du}/_{dx} = -\sin x$. By substitution,

$$I = \int u^4 \sin x \times \frac{du}{-\sin x} = -\int u^4\, du = -\frac{u^5}{5} + C\,.$$

(b) $\int \sqrt{1-x^2}\, dx$
Let $x = \sin\alpha$. Then ${}^{dx}/_{d\alpha} = \cos\alpha$. By substitution,

$$I = \int \sqrt{1-\sin^2\alpha}\, \cos\alpha\, d\alpha = \int \cos^2\alpha\, d\alpha = \int \frac{1+\cos 2\alpha}{2}\, d\alpha = \frac{1}{2}\left(\alpha + \frac{\sin 2\alpha}{2}\right) + C\,.$$

We need to convert back to x. We know that $x = \sin\alpha$, so $\alpha = \sin^{-1} x$. That gives us

$$I = \frac{1}{2}\left(\sin^{-1} x + \frac{\sin\left(2\sin^{-1} x\right)}{2}\right) + C\,.$$

To simplify the second term, first use a double-angle identity:

$$I = \frac{1}{2}\left(\sin^{-1} x + \frac{2\sin\left(\sin^{-1} x\right)\cos\left(\sin^{-1} x\right)}{2} + C\right)\,.$$

Certainly $\sin\left(\sin^{-1} x\right) = x$. To simplify $\cos\left(\sin^{-1} x\right)$, look at it this way: we want to find $\cos\beta$ where $\beta = \sin^{-1} x$, or in other words, where $\sin\beta = {}^{x}/_{1}$. Using right-angle trigonometry, we know the side opposite β is x, and the hypotenuse is 1. We want $\cos\beta$, so we need the side adjacent to β; from the Pythagorean Theorem, that is $\sqrt{1-x^2}$, so $\cos\beta = \sqrt{1-x^2}/1 = \sqrt{1-x^2}$. Hence

$$I = \frac{1}{2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right) + C\,.$$

(Notice that the 2's cancel in the second term.)

(c) $\displaystyle\int \frac{3}{x+x^3}\,dx$

Use partial fraction decomposition:

$$\frac{3}{x+x^3} = \frac{3}{x\left(1+x^2\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$3 = A\left(x^2+1\right) + (Bx+C)\,x$$

When $x = 0$, this equation simplifies to $3 = A$. To find B and C, we substitute $x = \pm 1$:

$$\begin{aligned} x = 1: &\quad 3 = 3\times 2 + B + C \\ x = -1: &\quad 3 = 3\times 2 + B - C \end{aligned}$$

We can simply subtract these equations to see that

$$0 = 0 + 2C\,,$$

so $C = 0$. Back-substitute to find that

$$3 = 3\times 2 + B + 0 \quad\Longrightarrow\quad B = -3\,.$$

Hence

$$I = \int \frac{3}{x} - \frac{3x}{x^2+1}\,dx = 3\ln|x| - 3\int \frac{x}{x^2+1}\,dx\,.$$

To simplify this last integral, let $u = x^2+1$, so that $dx = du/(2x)$, and

$$I = 3\ln|x| - 3\int \frac{x}{u}\frac{du}{2x} = 3\ln|x| - \frac{3}{2}\int\frac{du}{u} = 3\ln|x| - \frac{3}{2}\ln\left(x^2+1\right) + C\,.$$

(d) $\int e^{2x}\sin x\,dx$

This looks like a product, so we will integrate by parts. Let

$$\begin{aligned} u &= e^{2x} & v' &= \sin x \\ u' &= 2e^{2x} & v &= -\cos x\,. \end{aligned}$$

Then

$$I = uv - \int u'v\,dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx\,.$$

It looks as if we need to integrate by parts again. Let

$$\begin{aligned} u &= e^{2x} & v' &= \cos x \\ u' &= 2e^{2x} & v &= \sin x \end{aligned}$$

Then

$$I = -e^{2x}\cos x + 2\left(uv - \int u'v\,dx\right) = -e^{2x}\cos x + 2\left(e^{2x}\sin x - 2\int e^{2x}\sin x\,dx\right)\,.$$

We seem to be going in circles, but that's not really a problem, because we we can substitute I and solve a linear equation:

$$\begin{aligned} I &= -e^{2x}\cos x + 2e^{2x}\sin x - 4I \\ 5I &= -e^{2x}\cos x + 2e^{2x}\sin x \\ I &= \frac{1}{5}\left(-e^{2x}\cos x + 2e^{2x}\sin x\right) + C \; . \end{aligned}$$

(e) $\displaystyle\int \tan^3 x \sec x \, dx$

Both degrees are odd, so isolate $\sec x \tan x$ and use the Pythagorean identities to rewrite everything else as a power of secant:

$$I = \int \underbrace{\tan^2 x}_{\text{“everything else”}} \sec x \tan x \, dx = \int \left(\sec^2 x - 1\right) \sec x \tan x \, dx \; .$$

Let $u = \sec x$, so that $du/dx = \sec x \tan x$, as we have

$$I = \int \left(u^2 - 1\right) \cancel{\sec x \tan x} \frac{du}{\cancel{\sec x \tan x}} = \frac{u^3}{3} - u = \frac{\sec^3 x}{3} - \sec x + C \; .$$

(f) $\displaystyle\int \cos^4 x \sin^2 x \, dx$

Both powers are even, so rewrite using half-angle formulas:

$$\begin{aligned} I &= \int \left(\cos^2 x\right)^2 \sin^2 x \, dx \\ &= \int \left(\frac{1+\cos 2x}{2}\right)^2 \left(\frac{1-\cos 2x}{2}\right) dx \\ &= \frac{1}{8}\int \left(1 + 2\cos 2x + \cos^2 2x\right)\left(1 - \cos 2x\right) \, dx \\ &= \frac{1}{8}\int 1 + \cos 2x - \cos^2 2x - \cos^3 2x \, dx \; . \end{aligned}$$

I will split this into three integrals:

$$I_1 = \int 1 + \cos 2x \, dx \quad I_2 = \int \cos^2 2x \, dx \quad I_3 = \int \cos^3 2x \, dx \; .$$

The first integral is easy:

$$I_1 = x + \frac{\sin 2x}{2} \; .$$

The second integral requires another half-angle formula:

$$I_2 = \int \frac{1+\cos 4x}{2} \, dx = \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right) \; .$$

The third integral requires us to isolate $\cos 2x$ and use a Pythagorean identity to rewrite everything else as a power of secant:

$$I_3 = \int \underbrace{\cos^2 2x}_{\text{everythign else}} \cos 2x\, dx = \int \left(1 - \sin^2 2x\right) \cos 2x\, dx\ .$$

Let $u = \sin 2x$, so that ${du}/{dx} = 2\cos 2x$, and we have

$$I_3 = \int \left(1 - u^2\right) \cos 2x \frac{du}{2\cos 2x} = \frac{1}{2} \int 1 - u^2\, du = \frac{1}{2}\left(u - \frac{u^3}{3}\right) = \frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right)\ .$$

Put it all together, and

$$I = \frac{1}{8}\left[\underbrace{\left(x + \frac{\sin 2x}{2}\right)}_{I_1} - \underbrace{\frac{1}{2}\left(x + \frac{\sin 4x}{4}\right)}_{I_2} - \underbrace{\frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right)}_{I_3}\right] + C\ .$$

(g) $\displaystyle\int_{\pi/4}^{\pi/4} \tan^{55} x\, dx$

Don't try doing this by hand, but rather think of the geometry. The value is 0.

(h) $\displaystyle\int_0^{\pi} \tan x\, dx$

Careful! This is an improper integral! There is an asymptote at $x = {\pi}/{2}$. Separate it as

$$I = \lim_{b \to \frac{\pi}{2}^-} \int_0^b \tan x\, dx + \lim_{a \to \frac{\pi}{2}^+} \int_a^{\pi} \tan x\, dx\ .$$

To integrate $\tan x$ we use a trigonometric identity:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = \int \frac{\sin x}{u} \frac{du}{-\sin x} = -\int \frac{du}{u} = -\ln|\tan x|\ .$$

Hence

$$\begin{aligned} I &= \lim_{b \to \frac{\pi}{2}^-} -\ln|\cos x|\Big|_0^b + \lim_{a \to \frac{\pi}{2}^+} -\ln|\cos x|\Big|_a^{\pi} \\ &= \lim_{b \to \frac{\pi}{2}^-} \left(-\ln|\cos b| + \ln|\cos 0|\right) + \lim_{a \to \frac{\pi}{2}^+} \left(-\ln|\cos \pi| + \ln|\cos a|\right)\ . \end{aligned}$$

However, as $b \to ({\pi}/{2})^-$, $\cos b \to 0$, and then $\ln|\cos b|$ diverges, so at least one limit diverges, so the integral diverges.

This is a really evil integral, too, since if you don't think about the asymptote it converges.

(i) $\displaystyle\int_0^{\pi} \tan^2 x\, dx$

Same consideration (asymptote in the interval). Separate it as

$$I = \lim_{b \to \frac{\pi}{2}^-} \int_0^b \tan^2 x\, dx + \lim_{a \to \frac{\pi}{2}^+} \int_a^{\pi} \tan^2 x\, dx\ .$$

To integrate $\tan^2 x$ we use a Pythagorean identity:

$$\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x \; .$$

Hence

$$\begin{aligned} I &= \lim_{b \to \frac{\pi}{2}^-} (\tan x - x)\big|_0^b + \lim_{a \to \frac{\pi}{2}^+} (\tan x - x)\big|_a^\pi \\ &= \lim_{b \to \frac{\pi}{2}^-} \left[(\tan b - b) - (\tan 0 - 0)\right] + \lim_{a \to \frac{\pi}{2}^+} \left[(\tan \pi - \pi) - (\tan a - a)\right] \; . \end{aligned}$$

However, as $b \to (\pi/2)^-$, $\tan b \to \infty$, so at least one limit diverges, so the integral diverges.
This is a really evil integral, too, since if you don't think about the asymptote it converges.

(j) $\displaystyle\int_0^\infty \frac{x}{x^2+1} \, dx$
Since the integral is improper, we rewrite it:

$$I = \lim_{b \to \infty} \int_0^b \frac{x}{x^2+1} \, dx \; .$$

To integrate, let $u = x^2 + 1$ so that $du/dx = 2x$, and then

$$\begin{aligned} I &= \lim_{b \to \infty} \int_{x=0}^b \frac{x}{u} \frac{du}{2x} \\ &= \frac{1}{2} \lim_{b \to \infty} \int_{x=0}^b \frac{du}{u} \\ &= \frac{1}{2} \lim_{b \to \infty} \ln |u| \Big|_{x=0}^b \\ &= \frac{1}{2} \lim_{b \to \infty} \ln \left|x^2 + 1\right| \Big|_{x=0}^b \\ &= \frac{1}{2} \lim_{b \to \infty} \left(\ln \left|b^2 + 1\right| - \ln 1\right) \; . \end{aligned}$$

Hwoever, as $b \to \infty$, $b^2 + 1 \to \infty$, and then $\ln (b^2+1) \to \infty$, so the limit diverges, and hence the integral diverges.

2.5. Decide the following questions.

(a) Does the area under the graph of $f(x) = xe^{-x^2}$ converge on the interval $[0, \infty)$? If so, what does it converge to? If not, why not?

We want to know if $\int_0^\infty xe^{-x^2}\,dx$ converges. Rewrite it as

$$I = \lim_{b\to\infty} \int_0^b xe^{-x^2}\,dx\,.$$

Let $u = -x^2$ so that ${}^{du}/_{dx} = -2x$ and then

$$\begin{aligned} I &= \lim_{b\to\infty} \int_{x=0}^b xe^u \frac{du}{-2x} \\ &= -\frac{1}{2} \lim_{b\to\infty} \int_{x=0}^b e^u\,du \\ &= -\frac{1}{2} \lim_{b\to\infty} e^u\big|_{x=0}^b \\ &= -\frac{1}{2} \lim_{b\to\infty} e^{-x^2}\Big|_0^b \\ &= -\frac{1}{2} \lim_{b\to\infty} \left(e^{-b^2} - e^0\right) \\ &= -\frac{1}{2}(0-1) \\ &= \frac{1}{2}\,. \end{aligned}$$

The interval converges to ${}^1/_2$.

(b) Does the area under the graph of $f(x) = \left(e^{1/x^2}\right)/x^3$ converge on the interval $[-1, 1]$? If so, what does it converge to? If not, why not?

We want to know if $\int_{-1}^1 \left(e^{1/x^2}\right)/x^3\,dx$ converges. We have to be careful here, because there is an asymptote at $x = 0$. Split the integral into two:

$$I = \lim_{b\to 0^-} \int_{-1}^b \frac{e^{1/x^2}}{x^3}\,dx + \lim_{a\to 0^+} \int_a^1 \frac{e^{1/x^2}}{x^3}\,dx\,.$$

To handle the integrals, let $u = {}^1/_{x^2}$, so that ${}^{du}/_{dx} = -{}^2/_{x^3}$, and then

$$\int \frac{e^{1/x^2}}{x^3}\,dx = \int \frac{e^u}{x^3} \times \frac{x^3\,du}{-2} = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u = -\frac{1}{2}e^{1/x^2}\,.$$

Hence

$$\begin{aligned} I &= \lim_{b\to 0^-} -\frac{1}{2}e^{1/x^2}\Big|_{-1}^b + \lim_{a\to 0^+} -\frac{1}{2}e^{1/x^2}\Big|_a^1 \\ &= \lim_{b\to 0^-}\left(-\frac{1}{2}e^{1/b^2} + \frac{1}{2}e^{1/(-1)^2}\right) + \lim_{a\to 0^+}\left(-\frac{1}{2}e^{1/1^2} + \frac{1}{2}e^{1/a^2}\right)\,. \end{aligned}$$

However, as $b \to 0^-$, then ${}^1/_{b^2} \to \infty$, so that $e^{1/b^2} \to \infty$. At least one of the limits diverges, so the integral diverges.

(c) Why do we say that $\int_{-1}^{1} 1/x \, dx$ diverges, when its graph is plainly symmetric and suggests an area of 0?

The asymptote at $x = 0$ means that we have to split it into two integrals:

$$\int_{-1}^{0} \frac{1}{x} \, dx + \int_{0}^{1} \frac{1}{x} \, dx \, .$$

At least one of these "sub-integrals" diverges (proof omitted; you should be able to verify it) so the original integral diverges. **We may not combine divergent integrals to obtain a convergent integral.**

2.7. Compute the following volumes, if they converge. If they do not, indicate that.

(a) The volume of the solid whose base is the region between $f(x) = 1/x^2$ and the x-axis over $[1, \infty)$ and whose cross-sections perpendicular to the x-axis are squares.

This requires volume by slicing. The slice at point x has a cross-section of a square of side length $1/x^2$, so the area of the cross-section is $\left(1/x^2\right)^2$. Hence

$$V = \int_{1}^{\infty} \left(\frac{1}{x^2}\right)^2 dx = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{x^4} \, dx = \lim_{b\to\infty} \left(-\frac{1}{3x^3}\right)\Bigg|_{1}^{b} = \lim_{b\to\infty} \left(-\frac{1}{3b^3} + \frac{1}{3 \times 1^3}\right) \to 0 + \frac{1}{3} = \frac{1}{3} \, .$$

(b) The volume of the solid formed by rotating the region between $f(x) = 1/x^2$ and the x-axis over $[1, \infty)$ about the x-axis.

This requires volume by discs. The radius is $f(x) = 1/x^2$. Hence

$$V = \pi \int_{1}^{\infty} \left(\frac{1}{x^2}\right)^2 dx = \pi \int_{1}^{\infty} \frac{1}{x^4} \, dx = \pi \times \frac{1}{3} = \frac{\pi}{3} \, .$$

(I used the fact that we had already computed the integral in part (a).)

(c) The volume of the solid formed by rotating the region between $f(x) = 1/x^2$ and the x-axis over $[1, \infty)$ about the y-axis.

This requires either volume by shells, or else rewriting the equation in terms of y. If we use shells, the height of each shell is $f(x) = 1/x^2$. Hence

$$V = 2\pi \int_{1}^{\infty} x \cdot \frac{1}{x^2} \, dx = 2\pi \int_{1}^{\infty} \frac{1}{x} \, dx \, .$$

You know from class that this integral diverges, so the volume likewise diverges.

2.8. Let $G(x) = \int_0^x \sin \pi t \, dt$.

(a) Evaluate $G(0)$. Explain how you determined your answer.

$G(0) = \int_0^0 \sin \pi t \, dt = 0$ because the interval is just a point.

(b) Use the Midpoint Rule to approximate $G(4)$, using 8 subintervals. Round your answer to the nearest thousandth.

We have $\Delta x = {}^{(4-0)}/_8 = {}^1/_2$. The subintervals' endpoints are $x_0 = 0$, $x_1 = {}^1/_2$, $x_2 = 1$, $x_3 = {}^3/_2$, $x_4 = 2$, $x_5 = {}^5/_2$, $x_6 = 3$, $x_7 = {}^7/_2$, $x_8 = 4$. The Midpoint Rule tells us

$$\begin{aligned}
G(4) &\approx \sum_{i=1}^{8} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x \\
&= \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_4}{2}\right)\right. \\
&\qquad \left. + f\left(\frac{x_4+x_5}{2}\right) + f\left(\frac{x_5+x_6}{2}\right) + f\left(\frac{x_6+x_7}{2}\right) + f\left(\frac{x_7+x_8}{2}\right)\right] \times \underbrace{\frac{1}{2}}_{\Delta x} \\
&= \left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right)\right. \\
&\qquad \left. + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \times \frac{1}{2} \\
&= \left(\sin\frac{\pi}{4} + \sin\frac{3\pi}{4} + \sin\frac{5\pi}{4} + \sin\frac{7\pi}{4} + \sin\frac{9\pi}{4} + \sin\frac{11\pi}{4} + \sin\frac{13\pi}{4} + \sin\frac{15\pi}{4}\right) \times \frac{1}{2} \\
&= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \times \frac{1}{2} \\
&= 0\,.
\end{aligned}$$

(c) Use Simpson's Rule to approximate $G(4)$, using 8 subintervals. Round your answer to the nearest thousandth.

We have $\Delta x = {}^{(4-0)}/_8 = {}^1/_2$. The subintervals' endpoints are $x_0 = 0$, $x_1 = {}^1/_2$, $x_2 = 1$, $x_3 = {}^3/_2$, $x_4 = 2$, $x_5 = {}^5/_2$, $x_6 = 3$, $x_7 = {}^7/_2$, $x_8 = 4$. Simpson's Rule tells us

$$\begin{aligned}
G(4) &\approx [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) \\
&\qquad + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8)] \times \frac{\Delta x}{3} \\
&= \left[\sin 0 + 4\sin\frac{\pi}{2} + 2\sin\pi + 4\sin\frac{3\pi}{2} + 2\sin 2\pi\right. \\
&\qquad \left. + 4\sin\frac{5\pi}{2} + 2\sin 3\pi + 4\sin\frac{7\pi}{2} + \sin 4\pi\right] \times \frac{{}^1/_2}{3} \\
&= (0 + 4 + 0 - 4 + 0 + 4 + 0 - 4 + 0) \times \frac{1}{6} \\
&= 0\,.
\end{aligned}$$

(d) Using the formulas

$$E_S \leq \frac{k\,(b-a)}{180}\,(\Delta x)^4\ ,$$

compute the upper bound for the error of Simpson's approximation. Comment on how this estimate compares to the absolute error.

First we need $f^{(4)}(x)$, the fourth derivative of f. The derivatives are

$$\begin{aligned} f'(x) &= \pi\cos\pi t \\ f''(x) &= -\pi^2\sin\pi t \\ f'''(x) &= -\pi^3\cos\pi t \\ f^{(4)}(x) &= \pi^4\sin\pi t\ . \end{aligned}$$

The maximum value on $[0,4]$ of its absolute value is $k=\pi^4$. Hence

$$E_S \leq \frac{\pi^4\cdot(4-0)}{180}\left(\frac{4-0}{8}\right)^4 = \frac{4\pi^4}{180}\times\frac{256}{4096} = \pi^4/720 \approx 0.1353\ .$$

To determine the absolute error, we need to know the actual value of $G(4)$, which in this case we can actually compute:

$$G(4) = \int_0^4 \sin\pi t\,dt = -\frac{\cos\pi t}{\pi}\bigg|_0^4 = -\frac{1}{\pi}(\cos 4\pi - \cos 0) = -\frac{1}{\pi}(1-1) = 0\ .$$

Our approximation was also 0 (in both cases, in fact). Hence the absolute error is 0, which is in fact smaller than the estimate error 0.1353.

(e) Do you expect the error for the Midpoint Rule to be more, or less, than E_S? Why or why not?

Ordinarily we'd expect the error for the Midpoint Rule to be less, but in this case the error is actually the same, because both approximations lucked out to give us the exact value.