Integral

September 11, 2017

Find or estimate $\int_{-2}^{2} x^3 - x \; dx$ using

(a) geometry,

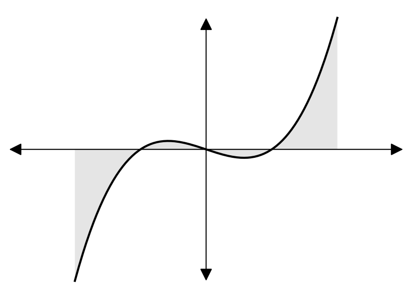
(b) four left endpoints,

(c) four right endpoints, and

(d) the definition.

Solution:

(a) From the geometry, we see that the curve is symmetric in such a way that the area of the regions above and below the x-axis cancel. Hence $\int_{-2}^{2} x^3 - x \; dx = 0$.



(b) Sample points for left endpoints are defined as

$$x_i^* = a + (i-1)\,\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n} \; .$$

With $a = -2$, $b = 2$, and $n = 4$, we have

$$\Delta x = \frac{2 - (-2)}{4} = 1 \quad \text{and} \quad x_i^* = -2 + (i - 1) \cdot 1 = i - 3 \ .$$

Hence

$$\begin{aligned} \int_{-2}^{2} f(x) \ dx &\approx \sum_{i=1}^{n} f(x_i^*) \Delta x \\ &= [f(1-3) + f(2-3) + f(3-3) + f(4-3)] \cdot 1 \\ &= f(-2) + f(-1) + f(0) + f(1) \\ &= -6 + 0 + 0 + 0 \\ &= -6 \ . \end{aligned}$$

(c) Sample points for right endpoints are defined as

$$x_i^* = a + i\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n} \ .$$

With $a = -2$, $b = 2$, and $n = 4$, we have

$$\Delta x = \frac{2 - (-2)}{4} = 1 \quad \text{and} \quad x_i^* = -2 + i \cdot 1 = i - 2 \ .$$

Hence

$$\begin{aligned} \int_{-2}^{2} f(x) \ dx &\approx \sum_{i=1}^{n} f(x_i^*) \Delta x \\ &= [f(1-2) + f(2-2) + f(3-2) + f(4-2)] \cdot 1 \\ &= f(-1) + f(0) + f(1) + f(2) \\ &= 0 + 0 + 0 + 6 \\ &= 6 \ . \end{aligned}$$

(d) For the definition, we will use right endpoints, which should be easiest. However, we no longer have a constant value of n, so we are looking at

$$\Delta x = \frac{b-a}{n} = \frac{2-(-2)}{n} = \frac{4}{n} \quad \text{and} \quad x_i^* = a + i\Delta x = -2 + \frac{4i}{n} \ .$$

By definition,

$$\begin{aligned}
\int_{-2}^{2} f(x)\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-2+\frac{4i}{n}\right)\cdot\frac{4}{n} \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left[\left(-2+\frac{4i}{n}\right)^3 - \left(-2+\frac{4i}{n}\right)\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left[\left(-8+\frac{48i}{n}-\frac{96i^2}{n^2}+\frac{64i^3}{n^3}\right)+\left(2-\frac{4i}{n}\right)\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(-6+\frac{44i}{n}-\frac{96i^2}{n^2}+\frac{64i^3}{n^3}\right) \\
&= \lim_{n\to\infty} \frac{4}{n} \left[\sum_{i=1}^{n}(-6)+\frac{44}{n}\sum_{i=1}^{n} i-\frac{96}{n^2}\sum_{i=1}^{n} i^2+\frac{64}{n^3}\sum_{i=1}^{n} i^3\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \left[-6n+\frac{44}{n}\times\frac{n(n+1)}{2}-\frac{96}{n^2}\times\frac{n(n+1)(2n+1)}{6}+\frac{64}{n^3}\times\frac{n^2(n+1)^2}{4}\right] \\
&= \lim_{n\to\infty} \left[-24+\frac{88n(n+1)}{n^2}-\frac{64n(n+1)(2n+1)}{n^3}+\frac{64n^2(n+1)^2}{n^4}\right] \\
&= -24+88-64\times 2+64 \\
&= 0\ .
\end{aligned}$$

("Typos" in class: neglected to multiply 4 to every term in the third line from the bottom, and the -6 changed to a 6 at one point.