TEST 3: IN CLASS

MAT 168

Directions: Solve as many problems as well as you can in the blue examination book, writing in pencil and showing all work. Put away any cell phones; the mere appearance will give a zero.

1. (50% of test) Compute **five** of the antiderivatives indicated. (Each is worth 10%.) Some require integration by u-substitution; others, integration by parts; still others, trigonometric techniques, including trigonometric substitution; and quite a few require multiple techniques.

(a) $\int \frac{\ln x}{x}\,dx$ (b) $\int \sec^4\theta \tan^2\theta\,d\theta$ (c) $\int x^5 \ln x\,dx$

(d) $\int \cos^4\alpha\,d\alpha$ (e) $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$ (f) $\int e^{-2x}\cos 5x\,dx$

(a) $\int \frac{\ln x}{x}\,dx$

Let $u = \ln x$. We have $du/dx = 1/x$, so $dx = x\,du$ and

$$I = \int \frac{u}{\cancel{x}}\cancel{x}\,du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C\,.$$

(b) $\int \sec^4\theta \tan^2\theta\,d\theta$

Since the secant's power is even, rewrite as

$$I = \int \sec^2\theta \cdot \sec^2\theta \tan^2\theta\,d\theta\,.$$

Use the Pythagorean identity to rewrite the second pair of secants:

$$I = \int \sec^2\theta \cdot \left(1 + \tan^2\theta\right)\tan^2\theta\,d\theta\,.$$

Now let $u = \tan\theta$. We have $du/d\theta = \sec^2\theta$, as expected (this is why we rewrote all but two secants), so $d\theta = d\theta/\sec^2\theta$ and

$$I = \int \cancel{\sec^2\theta} \cdot \left(1+u^2\right)u^2\,\frac{du}{\cancel{\sec^2\theta}} = \int u^2 + u^4\,du = \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + C\,.$$

(c) $\int x^5 \ln x\,dx$

Date: Fall 2017.

It's a product, and we don't see the derivative of either x^5 or of $\ln x$ in the integrand, so try integration by parts. Let $u = \ln x$ and $v' = x^5$. Then $u' = {}^1\!/_x$ and $v = {}^{x^6}\!/_6$. So

$$I = uv - \int u'v\,dx = \frac{x^6 \ln x}{6} - \int \frac{1}{x} \cdot \frac{x^6}{6}\,dx = \frac{x^6 \ln x}{6} - \int \frac{x^5}{6}\,dx = \frac{x^6 \ln x}{6} - \frac{x^6}{36} + C\,.$$

(d) $\displaystyle\int \cos^4 \alpha\,d\alpha$

Since the powers of the cosine and the (invisible) sine are both even, we have to use half-angle identities:

$$I = \int \left(\cos^2 \alpha\right)^2\,d\alpha = \int \left(\frac{1+\cos 2\alpha}{2}\right)^2\,d\alpha = \frac{1}{4}\int 1 + 2\cos 2\alpha + \cos^2 2\alpha\,d\alpha\,.$$

We have to apply the half-angle identity again to the last summand:

$$I = \frac{1}{4}\int 1 + 2\cos 2\alpha + \left(\frac{1+\cos 4\alpha}{2}\right)\,d\alpha\,.$$

Let's split this into three integrals to make life a little easier:

$$I = \frac{1}{4}\left(\int 1\,d\alpha + 2\int \cos 2\alpha\,d\alpha + \frac{1}{2}\int 1 + \cos 4\alpha\,d\alpha\right)\,.$$

We'll call the three integrals in the line above I_1, I_2, and I_3. The first integral should be easy: $I_1 = \int 1\,d\alpha = \alpha$. For the second integral, we need the substitution $u = 2\alpha$, so ${}^{du}\!/_{d\alpha} = 2$, so $d\alpha = {}^{du}\!/_2$, and we have

$$I_2 = \int \cos u\,\frac{du}{2} = \frac{\sin u}{2} + C = \frac{\sin 2\alpha}{2}\,.$$

The third integral also needs a substitution, $v = 4\alpha$, so $d\alpha = {}^{du}\!/_4$, and we have

$$I_3 = \int 1 + \cos u\,\frac{du}{4} = \frac{1}{4}(u + \sin u) + C = \alpha + \frac{\sin 4\alpha}{4}\,.$$

Putting them all together, we have

$$I = \frac{1}{4}\left[\alpha + 2\cdot\frac{\sin 2\alpha}{2} + \frac{1}{2}\left(\alpha + \frac{\sin 4\alpha}{4}\right)\right] + C = \frac{3\alpha}{8} + \frac{\sin 2\alpha}{4} + \frac{\sin 4\alpha}{32} + C\,.$$

(e) $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

We see the expression $9 - x^2$, which looks like $a^2 - x^2$, which should make us think of trigonometric substitution. Let $x = 3\sin\theta$; we have ${}^{dx}\!/_{d\theta} = 3\cos\theta$, so $dx = 3\cos\theta\,d\theta$.

That gives us

$$\begin{aligned} I &= \int \frac{(3\sin\theta)^2}{\sqrt{9-(3\sin\theta)^2}}\, 3\cos\theta\, d\theta \\ &= \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\, 3\cos\theta\, d\theta \\ &= \int \frac{9\sin^2\theta}{\sqrt{9\left(1-\sin^2\theta\right)}}\, 3\cos\theta\, d\theta \\ &= \int \frac{9\sin^2\theta}{\cancel{3\sqrt{\cos^2\theta}}}\, \cancel{3\cos\theta}\, d\theta\ . \end{aligned}$$

Since the powers of the (invisible) cosine and the sine are both even, we have to use half-angle identities:

$$I = 9\int \frac{1-\cos 2\theta}{2}\, d\theta\ .$$

We need the substitution $u = 2\theta$, so ${}^{du}/_{d\theta} = 2$, so $d\theta = {}^{du}/_{2}$. That gives us

$$I = \frac{9}{2}\int 1-\cos u\, \frac{du}{2} = \frac{9}{4}(u-\sin u)+C = \frac{9}{4}(2\theta - \sin 2\theta)+C\ .$$

This by itself will not do, because it's in terms of θ, and we need an answer in terms of x. It's easy to rewrite 2θ, since $\theta = \arcsin\left({}^{x}/_{3}\right)$. For $\sin 2\theta$, on the other hand, we use the double-angle formula:

$$\sin 2\theta = 2\sin\theta\cos\theta\ .$$

It's easy to rewrite $\sin\theta$, since $\sin\theta = {}^{x}/_{3}$. That leaves $\cos\theta$, for which we need to use right triangle properties. From $\sin\theta = {}^{x}/_{3}$, we know that there is a right triangle of hypotenuse length 3 whose side opposite θ will have length x. The side adjacent to θ will then have length $\sqrt{9-x^2}$, so $\cos\theta = {}^{\sqrt{9-x^2}}/_{3}$, and so

$$\sin 2\theta = 2\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} = \frac{2x\sqrt{9-x^2}}{9}\ .$$

Putting it all together, we have

$$I = \frac{9}{2}\arcsin\frac{x}{3} - \frac{x\sqrt{9-x^2}}{2} + C\ .$$

(f) $\int e^{-2x}\cos 5x\, dx$

It's a product, and we don't see the derivative of either e^{-2x} or $\cos 5x$ in the integrand, so try integration by parts. Let $u = e^{-2x}$ and $v' = \cos 5x$. We have $u' = -2e^{-2x}$ and $v = {}^{\sin 5x}/_{5}$, so

$$I = uv - \int u'v\, dx = \frac{e^{-2x}\sin 5x}{5} - \int \left(-2e^{-2x}\right)\left(\frac{\sin 5x}{5}\right)\, dx\ .$$

The new integral is again a product, and we don't see the derivative of either e^{-2x} or $\sin 5x$ in the integrand, so try integration by parts again. Let $u = e^{-2x}$ and $v' = \sin 5x$. We have $u' = -2e^{-2x}$ and $v = -\cos 5x/5$, so

$$\begin{aligned} I &= \frac{e^{-2x}\sin 5x}{5} + \frac{2}{5}\left[-\frac{e^{-2x}\cos 5x}{5} - \int \left(-2e^{-2x}\right)\left(-\frac{\cos 5x}{5}\right) dx\right] \\ &= \frac{e^{-2x}\sin 5x}{5} - \frac{2e^{-2x}\cos 5x}{25} - \frac{4}{25}\int e^{-2x}\cos 5x \, dx \\ &= \frac{e^{-2x}\sin 5x}{5} - \frac{2e^{-2x}\cos 5x}{25} - \frac{4}{25}\cdot I \, . \end{aligned}$$

We seem to be going in circles, but not really, as we can now solve for I:

$$\begin{aligned} \frac{29}{25}\cdot I &= \frac{e^{-2x}\sin 5x}{5} - \frac{2e^{-2x}\cos 5x}{25} \\ I &= \frac{25}{29}\left(\frac{e^{-2x}\sin 5x}{5} - \frac{2e^{-2x}\cos 5x}{25}\right) \\ &= \frac{5e^{-2x}\sin 5x}{29} - \frac{2e^{-2x}\cos 5x}{29} \, . \end{aligned}$$

2. (50% of test) Let

$$I = \int_0^\pi \sin x \, dx \, .$$

(a) (3%) Compute the exact value of I.
This should be straightforward:

$$I = -\cos x\big|_0^\pi = -(\cos\pi - \cos 0) = -(-1-1) = -(-2) = 2 \, .$$

(b) (5%) Approximate I using midpoint approximation and $n = 6$ subintervals.
We have $\Delta x = (\pi - 0)/6 = \pi/6$. The midpoints are $x_i^* = a + (i - 1/2)\,\Delta x$, so we have

$$\begin{aligned} I &\approx \sum_{i=1}^{6} f\left(x_i^*\right)\Delta x \\ &= \left(\sin\frac{\pi}{12} + \sin\frac{3\pi}{12} + \sin\frac{5\pi}{12} + \sin\frac{7\pi}{12} + \sin\frac{9\pi}{12} + \sin\frac{11\pi}{12}\right)\cdot\frac{\pi}{6} \\ &\approx 2.02303 \, . \end{aligned}$$

(c) (9%) Using the formula

$$E_M \le \frac{k\,(b-a)}{24}\,(\Delta x)^2 \, ,$$

compute the upper bound for the error of the midpoint approximation. Comment on how this estimate compares to the absolute error.

First we need to find $k \geq |f''(x)|$, where $f(x)$ is the integrand; that is, $f(x) = \sin x$. The second derivative of $\sin x$ is $-\sin x$, and the maximum value of $\sin x$ on $[0, \pi]$ is $y = 1$. So $k = 1$.[1] Thus

$$E_M \leq \frac{1 \cdot (\pi - 0)}{24} \cdot \left(\frac{\pi}{6}\right)^2 = \frac{\pi^3}{864} \approx .0359 \ .$$

The absolute error of our approximation was in fact

$$E_M = |2 - 2.02303| = .02303 < .0359 \ ,$$

so the absolute error is, as expected, smaller than the estimate.

(d) (5%) Approximate I using trapezoid approximation and $n = 6$ subintervals.
We have $\Delta x = \pi/6$. The endpoints of all intervals are $x = a + i\Delta x = i\pi/6$, so we have

$$\begin{aligned} I &\approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(x_6)\right] \\ &= \frac{\pi}{12}\left[\sin 0 + 2\sin\frac{\pi}{6} + 2\sin\frac{2\pi}{6} + 2\sin\frac{3\pi}{6} + 2\sin\frac{4\pi}{6} + 2\sin\frac{5\pi}{6} + \sin\frac{6\pi}{6}\right] \\ &\approx 1.95410 \ . \end{aligned}$$

(f) (9%) Using the formula

$$E_T \leq \frac{k(b-a)}{12}(\Delta x)^2 \ ,$$

compute the upper bound for the error of the trapezoid approximation. Comment on how this estimate compares to the absolute error.
Trapezoid approximation uses the same value of k as midpoint approximation, so

$$E_T \leq \frac{1 \cdot (\pi - 0)}{12} \cdot \left(\frac{\pi}{6}\right)^2 = \frac{\pi^3}{432} \approx .0718 \ .$$

The absolute error of our approximation was in fact

$$E_T = |2 - 1.95410| = 0.04590 < .0718 \ ,$$

so the absolute error is, as expected, smaller than the estimate.

(e) (5%) Approximate I using Simpson's Rule and $n = 6$ subintervals.
We have $\Delta x = \pi/6$. The endpoints of all intervals are $x = a + i\Delta x = i\pi/6$, so we have

$$\begin{aligned} I &\approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right] \\ &= \frac{\pi}{18}\left[\sin 0 + 4\sin\frac{\pi}{6} + 2\sin\frac{2\pi}{6} + 4\sin\frac{3\pi}{6} + 2\sin\frac{4\pi}{6} + 4\sin\frac{5\pi}{6} + \sin\frac{6\pi}{6}\right] \\ &\approx 2.00086 \ . \end{aligned}$$

[1] **Don't** just check the endpoints. It's the maximum value **on** the interval, so you have to think about the function's behavior over the entire interval. Sometimes the interval's maximum occurs at an endpoint but not usually.

(f) (9%) Using the formula

$$E_S \leq \frac{k(b-a)}{180}(\Delta x)^4 ,$$

compute the upper bound for the error of the midpoint approximation. Comment on how this estimate compares to the absolute error.

First we need to find $k \geq \left|f^{(4)}(x)\right|$, where $f(x)$ is the integrand; that is, $f(x) = \sin x$. The fourth derivative of $\sin x$ is again $\sin x$, and the maximum value of $\sin x$ on $[0,\pi]$ is $y = 1$. So $k = 1$.[2] Thus

$$E_S \leq \frac{1\cdot(\pi-0)}{180}\cdot\left(\frac{\pi}{6}\right)^4 = \frac{\pi^5}{233280} \approx .0013 .$$

The absolute error of our approximation was in fact

$$E_S = |2 - 2.00086| = .00086 < .0013 ,$$

so the absolute error is, as expected, smaller than the estimate.

(g) (5%) Find the value of n that guarantees $E_S < 10^{-3}$.

Again, we use $k = 1$, but now we want to find n, which means we don't know Δx. All we can do is substitute $\Delta x = (\pi-0)/n$, and solve

$$10^{-3} \leq \frac{1\cdot(\pi-0)}{180}\cdot\left(\frac{\pi}{n}\right)^4 = \frac{\pi^5}{180n^4} .$$

Isolate n, obtaining

$$10^{-3}\cdot 180n^4 \leq \pi^5 \quad\Longrightarrow\quad n^4 \leq \frac{\pi^5}{10^{-3}180} \quad\Longrightarrow\quad n \leq \sqrt[4]{\frac{\pi^5}{10^{-3}180}} \approx 6.4 .$$

This means we have to use more than $n = 6$ subintervals. Simpson's Rule requires an even number of subintervals, so we skip over $n = 7$ and conclude with $n = 8$.

[2] I made $f(x)$ a little easy to differentiate for this problem. You should expect $f(x)$ on the final to be somewhat harder; in particular, expect to use the chain rule.