DEFINITIONS/BIG-TIME FACTS TO KNOW FOR TEST 2

DEFINITIONS

All previous definitions, especially:
(definite) integral of $f(x)$ over $[a,b]$
(geometric) the *net* area between the curve of $f(x)$ and the x-axis, starting at $x=a$ and ending at $x=b$
(algebraic) $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$, where $\Delta x = {}^{b-a}/_{n}$ and x_i^* is *any* point in the ith subinterval of width Δx of $[a,b]$, *as long as the limit exists*

But also:
antiderivative of $f(x)$: any function F such that $F'(x)=f(x)$
(indefinite) integral of $f(x)$: an antiderivative of f

Applications of integrals: If the value V of a real-world phenomenon can be approximated by dividing an interval $[a,b]$ into n subintervals and adding the values of another function f on those subintervals, so that $V\approx\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$, then we can eliminate error by taking the limit as $n\to\infty$, so that the value of V is $\int_a^b f(x)\,dx$.

You should be able to use this principle to explain any application. For instance, were I to ask, "Why is the formula for the area under a function f given as $A=\int_a^b f(x)\,dx$?" the best answer would be,

> We can approximate the area under f on $[a,b]$ by dividing $[a,b]$ into n subintervals and adding the areas of rectangles whose width is $\Delta x = {}^{(b-a)}/_{n}$ and whose height is $f\left(x_i^*\right)$, where x_i^* is a sample point on each subinterval. That is,
> $$A\approx\sum_{i=1}^{n}\ell_i w_i=\sum_{i=1}^{n} f(x_i^*)\Delta x\ .$$
> To eliminate the error, let the number of rectangles approach ∞, so that
> $$A=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x=\int_a^b f(x)\ dx\ .$$

You wouldn't need such a detailed answer for full credit; the main point is that I see division of an interval, a sensible explanation of the function used to approximate the quantity, and a reference to the limit giving us the integral. Naturally, the more detailed and correct your answer is, the more likely you receive full credit.

Be ready to apply this principle to *any* of the following applications:

area between two curves	average value of a function	net change of a function
volume by slicing	volume by disks or washers	volume by shells

BIG-TIME RESULTS

Theorem (similar/related to 4.11 in the book)**.** *(A) If $f(x)$ and $g(x)$ have the same derivative, then f and g differ by a constant; that is, $f(x) = g(x) + C$ for some constant C.*
(B) If $F(x)$ is one antiderivative of $f(x)$, every *antiderivative of $f(x)$ has the form $F(x) + C$ for some constant C.*

Theorem (Fundamental Theorem of Calculus)**.** *Let f be continuous on $[a, b]$.*

(Part I) *The area function $A(x) = \int_a^x f(x)\,dx$ is continuous and differentiable on $[a, b]$, and its derivative is*

$$A'(x) = \frac{d}{dx} \int_a^x f(x)\ dx = f(x)\ .$$

(Part II) $\int_a^b f(x)\,dx = F(b) - F(a)$*, where F is* ***any*** *antiderivative of f.*

Theorem (The Chain Rule for Integrals; or, u-substitution)**.** *If f is a function of a variable u, which in turn is a function of x, and F is the antiderivative of f, then*

$$F(u) = \int f(u)\, u'\, dx = \int f(u)\, du.$$

Theorem (Mean Value Theorem for Integrals)**.**
(intuitive) If f is well-behaved on an interval, at least of its y-values is the function's average y-value.
(precise) If f is continuous on $[a, b]$, then we can find $c \in (a, b)$ such that $f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$.

INTEGRATION TABLE

I have left off "$+C$". Don't forget that.

$\int k\ dx = kx$ (where k is a constant)

$\int x^n\ dx = \dfrac{x^{n+1}}{n+1}$ (where $n \neq 1$)

$\int \dfrac{1}{x} = \ln|x|$

$\int e^x\ dx = e^x$

$\int a^x\ dx = \dfrac{a^x}{\ln a}$ (where $a > 0$ but $a \neq 1$)

$\int \sin x\ dx = -\cos x$ $\quad$ $\int \cos x\ dx = \sin x$

$\int \sec^2 x\ dx = \tan x$ $\quad$ $\int \sec x \tan x = \sec x$

$\int \csc^2 x\ dx = -\cot x$ $\quad$ $\int \csc x \cot x = -\csc x$

$\int \dfrac{1}{1+x^2}\, dx = \arctan x$ $\quad$ $\int \dfrac{1}{\sqrt{1-x^2}}\, dx = \arcsin x$

$$\int \frac{1}{x\sqrt{x^2-1}} = \text{arcsec}\ x$$

Integration Heuristic.	
Table	Is the integrand in the table?
Geometry	Does the integral represent an easily-found geometric value?
Algebra	Can I rewrite the integral by expanding or simplifying?
Substitution	Does the integral have an inside?

EXAMPLE PROBLEMS

This list is by no means exhaustive.

1. Compute the derivative of the following functions.

   (a) $F(x) = \int_3^x e^{-t^2}\,dt$ (b) $G(x) = \int_x^5 e^{-t^2}\,dt$ (c) $H(x) = \int_6^{x^2} e^{-t^2}\,dt$

2. Compute antiderivatives of the following functions.

   (a) $\sqrt[4]{x^3}$ (b) $\sec^2\theta + \csc^2\theta$ (c) $t^{-1} + t^{-2}$

   (d) $\dfrac{x^2 - 2x}{x^2}$ (e) $\cos 2\theta - \tan 3\theta$ (f) $\dfrac{3}{t^2+9}$

   (g) $\sin^2\theta$

3. For the following applied problems, (i) set up an integral to solve the problem, (ii) evaluate the integral, (iii) when appropriate, indicate the correct units, and (iv) when requested, provide the requested diagram.
   (a) Find the average value of $f(x) = 1/(3x)$ on the interval $[1,3]$. Diagram f and its average value in appropriate fashion.
   (b) If the marginal cost of producing n thousand widgets is $MC(n) = (n-3)^2 - 1$ thousands of dollars, determine the net change in cost to increase production from 1 thousand to 3 thousand widgets.
   (c) Find the total area of the region between the curves $f(x) = 4 - x$ and $g(x) = \sqrt{1-x^2}$ over the interval $[0,1]$. To evaluate the integral it may help to use geometry. Draw a diagram of the region.
   (d) Use the method of slicing to find the volume of the solid whose base is the region between $y = x^2$, the x-axis, and the line $x = 1$ and whose cross-sections perpendicular to the x-axis are squares.
   (e) Use the method of disks and washers to find the volume of the solid formed by rotating the region defined in part (c) about the x-axis.
   (f) Use the method of shells to find the volume of the solid formed by rotating the region defined in part (c) about the y-axis.

SOLUTIONS TO EXAMPLE PROBLEMS

1. This problem essentially asks you to apply the Fundamental Theorem of Calculus, Part I. **You need to state that you are applying the Fundamental Theorem of Calculus.** If you don't state that, you lose points.
   (a) $F'(x) = {}^d\!/_{dx} \int_3^x e^{-t^2}\,dt = e^{-x^2}$
   (b) $G'(x) = {}^d\!/_{dx} \int_x^5 e^{-t^2}\,dt = {}^d\!/_{dx}\left(-\int_5^x e^{-t^2}\,dt\right) = -{}^d\!/_{dx} \int_5^x e^{-t^2}\,dt = -e^{-x^2}$
   (c) $H'(x) = {}^d\!/_{dx} \int_6^{x^2} e^{-t^2}\,dt = e^{-\left(x^2\right)^2} \cdot \underbrace{2x}_{\text{chain rule}}$
2.
   (a) $\int \sqrt[4]{x^3}\,dx = \int x^{3/4}\,dx = {}^{x^{7/4}}\!/_{7/4} + C = {}^{4\sqrt[4]{x^7}}\!/_7 + C$
   (b) $\int \sec^2\theta + \csc^2\theta\,d\theta = \tan\theta - \cot\theta + C$
   (c) $\int x^{-1} + x^{-2}\,dx = \int {}^1\!/_x + x^{-2}\,dx = \ln|x| + {}^{x^{-1}}\!/_{-1} + C = \ln|x| - {}^1\!/_x + C$
   (d) $\displaystyle\int \frac{x^2-2x}{x^2}\,dx = \int 1 - {}^2\!/_x\,dx = \int 1\,dx - 2\int {}^1\!/_x\,dx = x - 2\ln|x| + C$
   (e) $\int \cos 2\theta - \tan 3\theta\,d\theta = \int \cos 2\theta\,d\theta - \int \tan 3\theta\,d\theta$
   These functions don't appear on our table, but they do resemble $\cos u$ and $\sin u$, so we try substitution. For the first integral let $u = 2\theta$; for the second, let $v = 3\theta$. We have

$$\begin{array}{ll} \dfrac{du}{d\theta} = 2 & \dfrac{dv}{d\theta} = 3 \\ \dfrac{du}{2} = d\theta & \dfrac{dv}{3} = d\theta \end{array} .$$

   The integrals become

$$\int \cos u \cdot \frac{du}{2} - \int \tan v \cdot \frac{dv}{3} = \frac{1}{2}\int \cos u\,du - \frac{1}{3}\int \tan v\,dv = \frac{1}{2}\sin u - \frac{1}{3}\int \tan v\,dv .$$

   But what of $\tan v$? This also doesn't show up on your table. *O tempora! O mores!*
   This is one of those cases where substitution shows up in disguise. In particular, $\tan v = {}^{\sin v}\!/_{\cos v}$. Let $w = \cos v$ and we have

$$\frac{dw}{dv} = -\sin v \quad \Longrightarrow \quad -\frac{dw}{\sin v} = dv .$$

   So

$$\frac{1}{3}\int \tan v\,dv = \frac{1}{3}\int \frac{\sin v}{\cos v}\,dv = \frac{1}{3}\int \frac{\cancel{\sin v}}{w}\cdot\left(-\frac{dw}{\cancel{\sin v}}\right) = -\frac{1}{3}\int \frac{1}{w}\,dw$$
$$= -\frac{1}{3}\ln|w| = \ln\left(\frac{1}{\sqrt[3]{|w|}}\right) = \ln\left(\frac{1}{\sqrt[3]{|\cos v|}}\right) .$$

So the original integral evaluates to

$$\frac{1}{2}\sin u - \ln\left(\frac{1}{\sqrt[3]{|\cos v|}}\right) + C = \frac{1}{2}\sin 2\theta - \ln\left(\frac{1}{\sqrt[3]{|\cos 3\theta|}}\right) + C\ .$$

We can also write this as

$$\frac{1}{2}\sin 2\theta + \ln\sqrt[3]{|\cos 3\theta|} + C\ .$$

(f) $\int \frac{3}{t^2+9}\,dt = 3\int \frac{1}{t^2+9}\,dt$

This one's also not in our table, but it does resemble $1/1+u^2$. Unfortunately, the 9 is a little problematic. To fix it, multiply numerator and denominator by $1/9$:

$$3\int \frac{1}{t^2+9}\cdot\frac{1/9}{1/9}\,dt = (3\cdot 1/9)\int \frac{1}{\frac{t^2}{9}+\frac{9}{9}}\,dt = \frac{1}{3}\int \frac{1}{(t/3)^2+1}\,dt\ .$$

This still resembles $1/1+u^2$, and the constant is now correct, so we try substitution. Let $u = t/3$ and we have

$$\frac{du}{dt} = \frac{1}{3} \quad\Longrightarrow\quad 3du = dt\ .$$

The integral becomes

$$\frac{1}{3}\int \frac{1}{u^2+1}\cdot 3du = \left(\frac{1}{3}\cdot 3\right)\int \frac{1}{u^2+1}\,du = \arctan u + C = \arctan\frac{t}{3} + C\ .$$

(Yes, you have to show the substitution. If you already know a shorter formula for this, you may not use it.)

(g) $\int \sin^2\theta\,d\theta$

Yet another integral that doesn't appear on the table. It looks a little like both $\int \sin u\,du$ and $\int u^2\,du$, but neither of those will help. You need instead to make use of the half-angle formula:

$$\int \sin^2\theta\,d\theta = \int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{1}{2}\int 1-\cos 2\theta\,d\theta\ .$$

Integrating 1 is trivial, but integrating $\cos 2\theta$ requires substitution. Let $u = 2\theta$ and we have

$$\frac{du}{d\theta} = 2 \quad\Longrightarrow\quad \frac{du}{2} = d\theta\ .$$

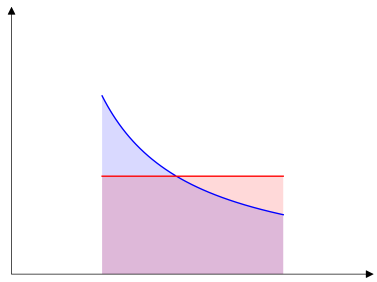
Hence

$$\frac{1}{2}\int 1-\cos 2\theta\,d\theta = \frac{1}{2}\int (1-\cos u)\cdot\frac{du}{2} = \left(\frac{1}{2}\cdot\frac{1}{2}\right)\int 1-\cos u\,du$$
$$= \frac{1}{4}(u-\sin u) + C = \frac{1}{4}(2\theta - \sin 2\theta) + C\ .$$

3.

(a) Average value is

$$\begin{aligned}\overline{y} &= \frac{1}{3-1}\int_1^3 \frac{1}{3x}\,dx = \left(\frac{1}{2}\cdot\frac{1}{3}\right)\int_1^3 \frac{1}{x}\,dx = \frac{1}{6}\left(\ln|x|\right)\Big|_1^3 \\ &= \frac{1}{6}(\ln 3 - \ln 1) = \frac{1}{6}(\ln 3 - 0) = \frac{1}{6}\ln 3 = \ln\sqrt[6]{3}\,.\end{aligned}$$



The line is $y = \ln\sqrt[6]{3}$; the curve is $y = {}^1\!/_{3x}$.

(b) Net change in cost is

$$C(b) - C(a) = \int_a^b C'(x)\;dx\,.$$

We can use marginal cost as an approximation to $C'(x)$, so

$$\begin{aligned}C(3) - C(1) &= \int_1^3 (n-3)^2 - 1\;dn = \int_1^3 n^2 - 6n + 8\;dn \\ &= \left(\frac{n^3}{3} - 3n^2 + 8n\right)\Bigg|_1^3 \\ &= \left[\left(\frac{27}{3} - 27 + 24\right) - \left(\frac{1}{3} - 3 + 8\right)\right] \\ &= \frac{2}{3}.\end{aligned}$$

The net change in cost would be roughly \$667.

(c) First we make sure there are no intersections to worry about:

$$\begin{aligned}4 - x &= \sqrt{1 - x^2} \\ 16 - 8x + x^2 &= 1 - x^2 \\ 2x^2 - 8x + 15 &= 0\,.\end{aligned}$$

This is a quadratic equation. We can solve for x using the quadratic formula:

$$x = \frac{8 \pm \sqrt{8^2 - 4\times 2\times 15}}{2\times 2} \approx \frac{8 \pm \sqrt{64 - 120}}{4}\,.$$

A negative in the square root implies that the roots are complex (have imaginary parts) so in fact there is no intersection. We proceed to computing the area:

$$A = \int_0^1 (4-x) - \sqrt{1-x^2}\, dx = \int_0^1 (4-x)\ dx - \int_0^1 \sqrt{1-x^2}\, dx\ .$$

The first integral is straightforward:

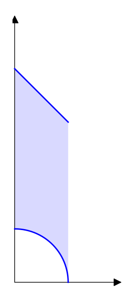
$$\int_0^1 4 - x\ dx = \left(4x - \frac{x^2}{2}\right)\Big|_0^1 = \left[\left(4\cdot 1 - \frac{1^2}{2}\right) - \left(4\cdot 0 - \frac{0^2}{2}\right)\right] = \frac{7}{2}\ .$$

The second integral is algebraically impossible for you at the moment, but if you recognize that $\sqrt{1-x^2}$ comes from a circle of radius 1 at the original, and the integral asks for the top-right *quarter* (not half!), then it's easy:

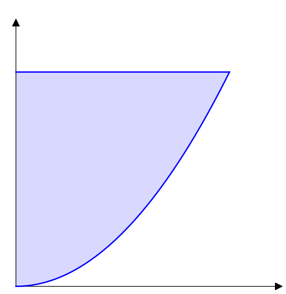
$$\int_0^1 \sqrt{1-x^2}\ dx = \frac{\pi\cdot 1^2}{4} = \frac{\pi}{4}\ .$$

So the area is

$$\frac{7}{2} - \frac{\pi}{4}\ .$$



(d) The base of the solid looks like this:



The cross sections perpendicular to the x-axis are squares with side length $s = 1 - x^2$. So the volume is

$$V = \int_0^1 B(x)\ dx = \int_0^1 s^2\ dx = \int_0^1 \left(1-x^2\right)^2\ dx = \int_0^1 1 - 2x^2 + x^4\ dx$$
$$= \left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)\Big|_0^1 = \left[\left(1 - \frac{2\cdot 1^3}{3} + \frac{1^5}{5}\right) - \left(0 - \frac{2\cdot 0^3}{3} + \frac{0^5}{5}\right)\right] = \frac{8}{15}\ .$$

(e) If we rotate the area in part (c) about the x-axis, the resulting solid of revolution will have volume

$$V = \pi \int_0^1 (4-x)^2 \, dx - \pi \int_0^1 \left(\sqrt{1-x^2}\right)^2 \, dx \, .$$

(The first integral is the outer volume, the second integral is the inner volume of the hole.) This is easy enough to integrate:

$$\begin{aligned}
V &= \pi \left[\int_0^1 16 - 8x + x^2 \, dx - \int_0^1 1 - x^2 \, dx \right] \\
&= \pi \left[\left(16x - 4x^2 + \frac{x^3}{3} \right)\Big|_0^1 - \left(x - \frac{x^3}{3} \right)\Big|_0^1 \right] \\
&= \pi \left[\left[\left(16 - 4 + \frac{1}{3} \right) - (0) \right] - \left[\left(1 - \frac{1}{3} \right) - (0) \right] \right] \\
&= \pi \left(12\frac{1}{3} - \frac{2}{3} \right) \\
&= \frac{35\pi}{3} \, .
\end{aligned}$$

(f) If we rotate the area in part (c) about the y-axis, the resulting solid of revolution will have volume

$$V = 2\pi \int_0^1 x \left[(4-x) - \sqrt{1-x^2} \right] \, dx \, .$$

This is easy enough to integrate, requiring only one substitution:

$$\begin{aligned}
V &= 2\pi \int_0^1 4x - x^2 - x\sqrt{1-x^2} \, dx \\
&= 2\pi \left(2x^2 - \frac{x^3}{3} + \frac{1}{2} \cdot \frac{2}{3} \cdot \left(\sqrt{1-x^2} \right)^3 \right)\Big|_0^1 \\
&= 2\pi \left[\left(2 - \frac{1}{3} + \frac{1}{3} \cdot 0 \right) - \left(0 - 0 + \frac{1}{3} \cdot 1 \right) \right] \\
&= 2\pi \left(\frac{5}{3} - \frac{1}{3} \right) \\
&= \frac{8\pi}{3} \, .
\end{aligned}$$