FINAL SET OF HOMEWORK SOLUTIONS (“QUIZ −7”)

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15. Truth be told, I don't like negative powers in the denominator, so if I can avoid them, I would. This is one of those cases, so I'd multiply numerator and denominator by e^x so as to obtain

$$\int \frac{e^x}{e^x - 2e^{-x}}\,dx = \int \frac{e^{2x}}{e^{2x} - 2}\,dx.$$

Let $u = e^{2x} - 2$ and you should be able to dispatch the rest.

On the other hand, you could also attack this by letting $u = e^x$. (This is probably what the author had in mind.) Notice $u' = e^x$, which appears in the numerator. The integral then becomes

$$I = \int \frac{1}{u - {}^2/_u}\,du = \int \frac{u}{u^2 - 2}\,du$$

and againyou should be able to dispatch the rest.

19. This strikes me as an evil sort of problem. Use trigonometric identities to rewrite the integral as

$$\int \frac{\cos^4 x}{\sin^6 x}\,dx = \int \left(\frac{\cos x}{\sin x}\right)^4 \frac{1}{\sin^2 x}\,dx = \int \cot^4 x \csc^2 x\,dx.$$

Let $u = \cot x$ and you should be able to dispatch the rest.

31. You have to divide $t^3 - 2$ by $t + 1$ first; this gives you $t^2 - t + 1 - {}^3/_{t+1}$. Hence

$$\int \frac{t^3 - 2}{t + 1}\,dt = \int t^2 - t + 1 - \frac{3}{t + 1}\,dt.$$

33. To complete the square: the coefficient of x^2 is 1, so we can simply halve the coefficient of x and add and subtract the square:

$$\int \frac{dx}{x^2 - 2x + 10} = \int \frac{1}{\left(x^2 - 2x + \left(\frac{2}{2}\right)^2\right) - \left(\frac{2}{2}\right)^2 + 10}\,dx = \int \frac{1}{(x^2 - 2x + 1) + 9}\,dx = \int \frac{1}{(x - 1)^2 + 9}\,dx.$$

Now let $u = x - 1$; $u' = 1$ so you should be able to dispatch the rest.

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19. Setting $v' = \tan^{-1} x$ would be futile, so the only path is to set $u = \tan^{-1} x$ and $v' = 1$. This gives us

$$u' = \frac{1}{x^2 + 1} \quad \text{and} \quad v = x.$$

The integral becomes

$$\int \tan^{-1} x\,dx = x \tan^{-1} x - \int \frac{x}{x^2 + 1}\,dx.$$

You should be able to dispatch the rest with a u-substitution for the latter integral.

23. (*Note: the t's accidentally fell over and became x's while I was typing this, but the solution is otherwise correct.*)

The guidelines suggest we want $u = x^2$ and $v' = e^{-x}$ (u should diminish when differentiated). This gives us

$$u' = 2x \quad \text{and} \quad v = -e^{-x}.$$

(Don't forget the chain rule when antidifferentiating v': it's not just x in the exponent!) The integral becomes

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2\int x e^{-x}\,dx.$$

The new integral is non-obvious, but we need not lose heart, for following the guidelines has ensured the integrand is in some sense "smaller." The guidelines suggest again that we want $u = x$ and $v' = e^{-x}$. This gives us

$$u' = 1 \quad \text{and} \quad v = -e^{-x}.$$

The integral becomes

$$-x^2 e^{-x} + 2\int x e^{-x}\,dx = -x^2 e^{-x} + 2\left(-x e^{-x} + \int e^{-x}\,dx\right).$$

You should be able to dispatch the rest.

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19. Don't let the radical scare you. Since the number of cosines is odd, rewrite all but one of them as sines:

$$\int \cos^3 x \sqrt{\sin x}\,dx = \int \cos x \left(1 - \sin^2 x\right) \sqrt{\sin x}\,dx.$$

Let $u = \sin x$; then $u' = \cos x$, which does indeed show up in the integral. The integral becomes

$$\int \underbrace{\cos x}_{u'} \left(1 - \sin^2 x\right) \sqrt{\sin x}\,dx = \int \left(1 - u^2\right) \sqrt{u}\,du.$$

You should be able to dispatch the rest.

31. The power of $\sec x$ is even, so set $u = \tan x$; then $u' = \sec x$, which does indeed show up in the integral. The integral becomes

$$\int 10 \tan^9 x \sec x\,dx = 10 \int u^9\,du.$$

You should be able to dispatch the rest.

39. One way to approach this problem is to rewrite it in terms of sine and cosine:

$$\int \frac{\csc^4 x}{\cot^2 x}\,dx = \int \frac{1/\sin^4 x}{\cos^2 x/\sin^2 x}\,dx = \int \frac{1}{\sin^4 x} \cdot \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1}{\sin^2 x} \cdot \frac{1}{\cos^2 x}\,dx = \int \csc^2 x \sec^2 x\,dx.$$

I will now approach this by parts. (Why? because it's a product, and integration by parts is precisely for the sake of products.) Set $u = \csc^2 x$ and $v' = \sec^2 x$. This gives us

$$u' = 2\csc x\left(-\csc x \cot x\right) \quad \text{and} \quad v = \tan x.$$

The integral becomes

$$\int \csc^2 x \sec^2 x\,dx = \csc^2 x \tan x - 2\int \csc x\left(-\csc x \cot x\right) \tan x\,dx.$$

At this point you're thinking I've lost my mind, and while you're right, that actually happened a long time ago. Lucky for us, my insanity has helped out, because as you should remember from trigonometry, $\cot x$ and $\tan x$ divide out, giving us a *very* convenient integral:

$$\csc^2 x \tan x - 2\int \csc x\,(-\csc x \cancel{\cot x})\cancel{\tan x}\,dx = \csc^2 x \tan x + 2\int \csc^2 x\,dx = \csc^2 x \tan x - 2\cot x + C.$$

Don't forget those C's.

The computer gives me a different-looking answer, $\tan x - \cot x + C$, but a little trigonometry shows these answers are in fact equivalent.

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35. As the hint suggests, you first need to complete the square in the denominator. However, the coefficient of x is negative, so first we need a little factoring:

$$\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{1}{\sqrt{3-(x^2+2x)}}\,dx = \int \frac{1}{\sqrt{3-(x^2+2x+1)+1}}\,dx = \int \frac{1}{\sqrt{4-(x+1)^2}}\,dx.$$

Let $u = x+1$; then $u' = 1$, so it is easy to rewrite the integral:

$$\int \frac{1}{\sqrt{4-(x+1)^2}}\,dx = \int \frac{1}{\sqrt{4-u^2}}\,du.$$

At this point, we have a form suitable for sine substitution. Let $u = 2\sin\theta$; then $u' = 2\cos\theta$, so the Chain Rule tells us that

$$\int \frac{1}{\sqrt{4-u^2}}\,du = \int \frac{1}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\,d\theta = \frac{1}{\cancel{\sqrt{4}}}\cdot\cancel{2}\int \frac{1}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\,d\theta = \int \frac{1}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\,d\theta = \theta.$$

Neat, hunh? Alas, we are not done, as we have to restore x. First we restore u using the fact that

$$u = 2\sin\theta \quad\Longrightarrow\quad \theta = \sin^{-1}\frac{u}{2}$$

and then we recall that $u = x+1$. So the integral has simplified to

$$\sin^{-1}\left(\frac{x+1}{2}\right) + C.$$

66. The area of the ellipse will be *twice* the value of the integral

$$\int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2}\,dx.$$

We have a form suitable for sine substitution. Let $x = a\sin\theta$; then $x' = a\cos\theta$, so the Chain Rule tells us that

$$\frac{b}{a}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}\int_{-\pi/2}^{\pi/2}\sqrt{a^2-(a\sin\theta)^2}\cdot a\cos\theta\,d\theta.$$

(The values of $-\pi/2$ and $\pi/2$ are the values of θ that solve $\pm a = a\sin\theta$, which we get from substituting $x = \pm a$ into the equation $x = a\sin\theta$.) The integral is now

$$\frac{b}{\cancel{a}}\cdot\sqrt{a^2}\cdot\cancel{a}\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta = ab\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta.$$

We solve this by the half-angle formula:

$$\begin{aligned} ab\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta &= ab\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta \\ &= \frac{ab}{2}\left(\theta+\frac{1}{2}\cos 2\theta\right)\Big|_{-\pi/2}^{\pi/2} \\ &= \frac{ab}{2}\left[\left(\frac{\pi}{2}+\cancel{\frac{1}{2}\cos\pi}\right)-\left(-\frac{\pi}{2}+\cancel{\frac{1}{2}\cos(-\pi)}\right)\right] \\ &= \frac{ab}{2}\pi. \end{aligned}$$

(The cosines cancel because cosine is an even function. Or, just calculate $\cos\pi = \cos(-\pi) = 0$.) Remember that we had to multiply this answer by 2, so the area is $ab\pi$.

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23. First we factor the denominator:

$$\frac{x^2+12x-4}{x^3-4x} = \frac{x^2+12x-4}{x(x+2)(x-2)} = \frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}.$$

Clear the denominators to find

$$x^2+12x-4 = A(x+2)(x-2)+Bx(x-2)+Cx(x+2).$$

If we let $x=0$, this equation simplifies to

$$-4 = A\cdot(-4),$$

so $A=1$. Let $x=2$ and $x=-2$ to find B and C. You should be able to dispatch the rest.

27. First we factor the denominator:

$$\frac{81}{x^3-9x^2} = \frac{81}{x^2(x-9)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-9}.$$

Clear the denominators to find

$$81 = Ax(x-9)+B(x-9)+Cx^2.$$

If we let $x=9$, this equation simplifies to

$$81 = 81C,$$

so $C=1$. If we let $x=0$, this equation simplifies to

$$81 = -9B,$$

so $B=-9$. To find A, let x be *any other* number, preferably an easy one, and substitute it into the equation, along with the values we found for B and C. I will pick $x=10$, because then $x-9=1$. (Clever, no?) The equation simplifies to

$$81 = 10A-9+100 \quad\Longrightarrow\quad -10 = 10A \quad\Longrightarrow\quad A=-1.$$

You should be able to dispatch the rest.

43. The denominator is already factored into irreducibles, so we want

$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

Clear the denominators to find

$$x^2+x+2 = A(x^2+1) + (Bx+C)(x+1).$$

If we let $x=-1$, this equation simplifies to

$$2 = 2A,$$

so $A=1$. If we let $x=0$, this equation simplifies to

$$2 = 1(0^2+1) + (0+C)(0+1),$$

so $C=1$. To find B, let x be *any other* number, preferably an easy one, and susbtitute it into the equation, along with the values we found for A and C. I will pick $x=1$ because it seems easy. The equation simplifies to

$$4 = 1\cdot 2 + (B+1)\cdot 2,$$

so $B=0$. You should be able to dispatch the rest.

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35. I will do only $n=8$. We have $f(x) = 3x^5 - 8x^3$, $a=0$, $b=4$, and $\Delta x = 4/8 = 1/2$, so

$$x_0 = 0,\quad x_1 = \frac{1}{2},\quad x_2 = 1,\quad x_3 = \frac{3}{2},\quad x_4 = 2,\quad \ldots,\quad x_7 = \frac{7}{2},\quad x_8 = 4.$$

Hence Simpson's Rule tells us

$$\int_a^b f(x)\,dx \approx \left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + \cdots + 4f\left(\frac{7}{2}\right) + f(4)\right]\frac{1/2}{3}$$
$$= 1537.$$

The absolute error is $E_S = |1537 - 1536| = 1$, which satisfies the estimate

$$E_S \le \frac{k(b-a)}{180}(\Delta x)^4 = \frac{1440(4-0)}{180}\left(\frac{1}{2}\right)^4 = 1.$$

You may be wondering how I came up with $k=1440$. Remember that this is determined by the value of $\left|f^{(4)}(x)\right|$ on the interval $[a,b]$. In this case the fourth derivative is $f^{(4)}(x) = 360x$, a linear function with positive slope. Its maximum occurs at the value $x=8$, so $k = f^{(4)}(8) = 1440$.

39(b) If the number of intervals in the Midpoint rule is increased by a factor of 3, the only aspect of the error inequality that changes is Δx; it changes from

$$\frac{b-a}{n} \quad \text{to} \quad \frac{b-a}{3n}.$$

Hence the error estimate changes from E_M to

$$E_{M'} = \frac{k(b-a)}{24}\cdot(\Delta x)^2 = \frac{k(b-a)}{24}\cdot\left(\frac{b-a}{3n}\right)^2 = \frac{k(b-a)^3}{216n^2}.$$

Compare this to the original:

$$E_M = \frac{k(b-a)}{24}\cdot(\Delta x)^2 = \frac{k(b-a)}{24}\cdot\left(\frac{b-a}{n}\right)^2 = \frac{k(b-a)^3}{24n^2}.$$

To determine the factor, take the ratio of the errors:

$$E_{M'}/E_M = \frac{\frac{k(b-a)^2}{216n^2}}{\frac{k(b-a)^2}{24n^2}} = \frac{\cancel{k(b-a)^2}}{216\cancel{n}} \times \frac{24\cancel{n}}{\cancel{k(b-a)^2}} = \frac{24}{216} = \frac{1}{9}.$$

As the answer is $1/9$, the error decreased by a factor of 9, not 8.

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29. This is Gabriel’s Horn, which I did in class Friday.

47. The function $\ln x^2$ has a vertical asymptote at $x = 0$, so we take the limit as $t \to 0$. I also exploit the power rule of logarithms to rewrite $\ln x^2 = 2\ln x$. The antiderivative of $\ln x$ can be found by integrating by parts; I believe it was an example in class, but if not, set $u = \ln x$ and $v' = 1$ to antidifferentiate. We have

$$\int_0^1 \ln x^2\,dx = \lim_{t\to 0^+} 2\int_t^1 \ln x\,dx = 2\lim_{t\to 0^+}\left[x(\ln x - 1)\right]\Big|_t^1 = 2\lim_{t\to 0^+}\left[\left(1\left(\overset{\nearrow 0}{\cancel{\ln 1}} - 1\right)\right) - t(\ln t - 1)\right].$$

You might think that latter term diverges, but amazingly enough you’d be wrong, which goes to emphasize how nonintuitive the world is! Notice that it has the form $0\times\infty$, and recall that this is an “indeterminate form” which, with a little algebraic massage, is content to submit t L’Hôpital’s Rule. To wit:

$$\lim_{t\to 0^+} t^{\nearrow 0}\left(\ln t^{\nearrow -\infty} - 1\right) = \lim_{t\to 0^+}\frac{\ln t - 1^{\nearrow -\infty}}{1/t_{\searrow \infty}} \overset{\text{L'H}}{=} \lim_{t\to 0^+}\frac{1/t}{-1/t^2} = \lim_{t\to 0^+}\left(\frac{1}{\cancel{t}}\times -\frac{t^{\cancel{2}}}{1}\right) = 0,$$

Substituting this back into our expression for the integral, we have

$$\int_0^1 \ln x^2\,dx = 2\lim_{t\to 0^+}\left[\left(1\left(\overset{\nearrow 0}{\cancel{\ln 1}} - 1\right)\right) - \cancel{t(\ln t - 1)}^{\nearrow 0}\right] = -2.$$