HOMEWORK QUIZ 6 SOLUTIONS

p. 432

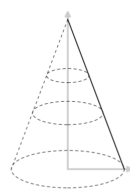
7. For shells, we use $V = 2\pi \int_a^b x f(x)\ dx$, so

$$V = 2\pi \int_0^2 \frac{x}{1+x^2}\, dx.$$

Let $u = 1 + x^2$; we need $u' = 2x$, which we actually have, so

$$V = \pi \int_{x=0}^{2} u^{-1}\, du = \pi \ln u\Big|_0^2 = \pi \ln\left(1+x^2\right)\Big|_0^2 = \pi\left(\ln 5 - \ln 1\right) = \pi \ln 5.$$

27. We can obtain a cone of radius 3 and height 8 by rotating the line $y = -\frac{8}{3}x + 8$ about the y-axis:



We can evaluate this volume by discs if we proceed along the y-axis, but the problem asks us to use shells, so the volume is

$$V = 2\pi \int_0^3 x\left(-\frac{8}{3}x + 8\right)\, dx = 2\pi\left(-\frac{8}{6}x^2 + 8x\right)\Bigg|_0^3 = 2\pi\left[\left(-\frac{8}{6}\cdot 9 + 24\right) - 0\right] = 24\pi.$$

This agrees with the standard formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\cdot\pi\cdot 3^2\cdot 8 = 24\pi$.

67. First off, note that only a moron would do the problem this way; this problem is much easier if you do it with discs. However, the book asks us to do it this way, so we are compelled to act like morons. Here we go:

We can view the sphere as a rotation of the circle $x^2 + (y-8)^2 = 64$. (I think it makes the problem easier *not* to place the sphere's center at the origin.) If we solve that equation for y, the *bottom* half of the sphere lies along

$$y = 8 - \sqrt{64 - x^2}.$$

Notice that when $x = 0$, $y = 0$, and when $x = 8$, $y = 8$; this agrees with placing the bowl's bottom at the origin. The water in the bowl rises to the y-value of h, so the height of each shell is

$$h - y = h - \left(8 - \sqrt{64 - x^2}\right) = (h - 8) + \sqrt{64 - x^2}.$$

For a height of $y = h$, the corresponding x-value is

$$x = \sqrt{64 - (h-8)^2}.$$

Again, notice that when $h = 0$, $x = 0$, and when $h = 8$, $x = 8$. This ugly expression makes the integral look messy, so we'll write b in its place until the end. The volume of water is thus

$$\begin{aligned} V &= 2\pi \int_0^b x\left[(h-8) + \sqrt{64 - x^2}\right]\,dx \\ &= 2\pi\left[\int_0^b (h-8)\,x\,dx + \int_0^b x\sqrt{64-x^2}\,dx\right]. \end{aligned}$$

Let $u = 64 - x^2$ for the second integral, and $u' = -2x$, so we have

$$\begin{aligned} V &= 2\pi\left[\frac{h-8}{2}\,x^2\Big|_0^b - \frac{1}{2}\int_{x=0}^b u^{1/2}\,du\right] \\ &= 2\pi\left[\frac{h-8}{2}\left(b^2 - 0\right) - \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3} \times u^{3/2}\Big|_{x=0}^b\right] \\ &= 2\pi\left[\frac{h-8}{2}\cdot b^2 - \frac{1}{3}\left[\left(64 - b^2\right)^{3/2} - \left(64 - 0^2\right)^{3/2}\right]\right] \\ &= 2\pi\left[\frac{h-8}{2}\cdot b^2 - \frac{1}{3}\left[\left(64-b^2\right)^{3/2} - 512\right]\right]. \end{aligned}$$

Notice at this point that, as per the book's directions, when $h = 0$ we have $b = 0$ and $V = 0$, and when $h = 8$ we have $b = 8$ and $V = {}^{1024\pi}/_3$, the correct volume for a hemisphere of radius 8. Of course, we want to substitute $b = \sqrt{64 - (h-8)^2}$, which gives us

$$\begin{aligned} V &= 2\pi\left[\frac{h-8}{2}\cdot\left(64 - (h-8)^2\right) - \frac{1}{3}\left[\left(\cancel{64} - \left(\cancel{64} - (h-8)^2\right)\right)^{3/2} - 512\right]\right] \\ &= 2\pi\left[\frac{h-8}{2}\left(64-(h-8)^2\right) - \frac{1}{3}\left[|h-8|^3 - 512\right]\right]. \end{aligned}$$

Notice a subtle point here: $\left((h-8)^2\right)^{3/2} = |h-8|^3$, and *not* $(h-8)^3$. This is because $h < 8$, so $h - 8 < 0$, so $(h-8)^3 < 0$, whereas the square makes $\left((h-8)^2\right)^{3/2} > 0$. We can alleviate this by reversing either h and 8 or the sign. It's more convenient to do the latter:

$$\begin{aligned} V &= 2\pi \left[\frac{h-8}{2}\left(64-(h-8)^2\right) - \frac{1}{3}\left[-(h-8)^3 - 512\right]\right] \\ &= 2\pi \left[32(h-8) - \frac{(h-8)^3}{2} + \frac{(h-8)^3}{3} + \frac{512}{3}\right] \\ &= \pi \left[64(h-8) - (h-8)^3 + \frac{2(h-8)^3}{3} + \frac{1024}{3}\right] \\ &= \pi \left[64(h-8) - \frac{3(h-8)^3}{3} + \frac{2(h-8)^3}{3} + \frac{1024}{3}\right] \\ &= \pi \left[64(h-8) - \frac{(h-8)^3}{3} + \frac{1024}{3}\right]. \end{aligned}$$

Notice that at $h = 0$, $V = 0$, and at $h = 8$, $V = 1024/3$, which is the correct volume for a hemisphere of radius 8. I realize the book has a prettier answer, but after all this work I don't care to figure out how they got it.

By way of comparison, solving the problem by discs is this easy:

$$\begin{aligned} V &= \pi \int_{8-h}^{8} \left(\sqrt{64-x^2}\right)^2 \, dx \\ &= \pi \int_{8-h}^{8} 64 - x^2 \, dx \\ &= \pi \left(64x - \frac{x^3}{3}\right)\Big|_{8-h}^{8} \\ &= \pi \left[\left(512 - \frac{512}{3}\right) - \left[64(8-h) - \frac{(8-h)^3}{3}\right]\right] \\ &= \pi \left[\frac{1024}{3} - 64(8-h) + \frac{(8-h)^3}{3}\right]. \end{aligned}$$

No fuss, no muss, and we still get the correct answer. It may *look* different, but a little magic with opposites and we have the original:

$$V = \pi \left[\frac{1024}{3} + 64(h-8) - \frac{(h-8)^3}{3}\right].$$

p. 440

9. Arclength is $s = \int_a^b \sqrt{1 + f'(x)^2}\, dx$, and if $f(x) = x^4/4 + 1/8x^2$, then $f'(x) = x^3 - 1/4x^3$, so

$$\begin{aligned}
s &= \int_1^2 \sqrt{1 + \left(x^3 - \frac{1}{4x^3}\right)^2}\, dx \\
&= \int_1^2 \sqrt{1 + \left(x^6 - \frac{1}{2} + \frac{1}{16x^6}\right)}\, dx \\
&= \int_1^2 \sqrt{x^6 + \frac{1}{2} + \frac{1}{16x^6}}\, dx \\
&= \int_1^2 \sqrt{\left(x^3 + \frac{1}{4x^3}\right)^2}\, dx \\
&= \int_1^2 x^3 + \frac{1}{4x^3}\, dx \\
&= \left(\frac{x^4}{4} - \frac{1}{8x^2}\right)\Bigg|_1^2 \\
&= \left(\frac{16}{4} - \frac{1}{32}\right) - \left(\frac{1}{4} - \frac{1}{8}\right) \\
&= \frac{127}{32} - \frac{1}{8} \\
&= \frac{123}{32}.
\end{aligned}$$

p. 449

15. The approximate volume to paint will be the product of the surface area and the thickness. Surface area is $S = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2}\, dx$, and if $f(x) = \sqrt{8x - x^2}$, then

$$f'(x) = \frac{8 - 2x}{2\sqrt{8x - x^2}} = \frac{4 - x}{\sqrt{8x - x^2}},$$

so

$$\begin{aligned}
S &= 2\pi \int_1^7 \sqrt{8x - x^2}\sqrt{1 + \left(\frac{4-x}{\sqrt{8x-x^2}}\right)^2}\,dx \\
&= 2\pi \int_1^7 \sqrt{8x - x^2}\sqrt{1 + \frac{(4-x)^2}{8x-x^2}}\,dx \\
&= 2\pi \int_1^7 \sqrt{8x - x^2} \cdot \sqrt{\frac{8x-x^2}{8x-x^2} + \frac{(4-x)^2}{8x-x^2}}\,dx \\
&= 2\pi \int_1^7 \sqrt{8x - x^2} \cdot \sqrt{\frac{\left(\cancel{8x - x^2}\right) + \left(16 \cancel{- 8x + x^2}\right)}{8x-x^2}}\,dx \\
&= 2\pi \int_1^7 \frac{\sqrt{8x-x^2}}{\sqrt{8x-x^2}} \cdot \sqrt{16}\,dx \\
&= 2\pi \int_1^7 4\,dx \\
&= 8\pi\, x\big|_1^7 \\
&= 48\pi.
\end{aligned}$$

As the surface area is $48\pi\mathrm{m}^2$, the volume of paint needed is

$$\begin{aligned}
48\pi\mathrm{m}^2 \times 1.5\mathrm{mm} &= 48\pi\mathrm{m}^2 \times \left(10^3 \mathrm{mm}/\mathrm{m}\right)^2 \times 1.5\mathrm{mm} = 48\pi \times 10^6 \times 1.5\ \mathrm{mm}^3 = 72\pi \times 10^6\ \mathrm{mm}^3 \\
&= 72\pi \times 10^6\ \mathrm{mm}^3 \times \left(10^{-3}\frac{\mathrm{m}}{\mathrm{mm}}\right)^3 = \frac{72\pi}{1000}\ \mathrm{m}^3 = \frac{9\pi}{125}\ \mathrm{m}^3.
\end{aligned}$$