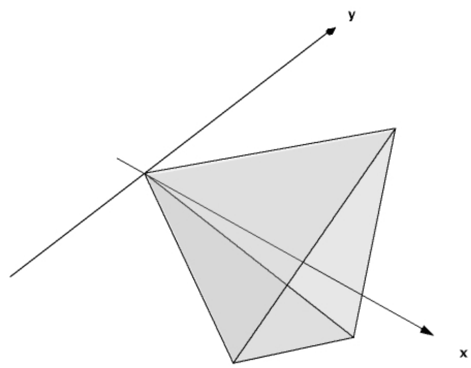
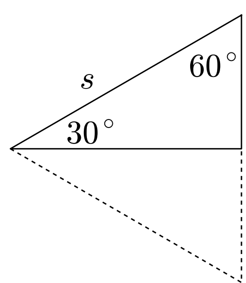
HOMEWORK QUIZ 5 SOLUTIONS (CORRECTION TO P. 420 #15)

p. 420

15. We can view a tetrahedron as an accumulation of equilateral triangles. We can view this as accumulation as we move along the x-axis from the origin to the height of the tetrahedron:



The area of an equilateral triangle comes from the fact that dropping an altitude turns it into a 30°-60°-90° triangle:

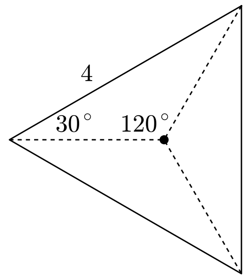


The base of the triangle has length s; the height has length $s\sqrt{3}/2$ (you can find this "easily" using the Pythagorean Theorem), so the area is $s^2\sqrt{3}/4$. The volume is thus

$$V = \int_0^h \frac{s^2\sqrt{3}}{4}\,dx.$$

We now need to do two things: find h, then express s as a function of x.

To find h, drop an altitude from the tip of the tetrahedron to its base. (This altitude runs along the x-axis.) Call the point where this altitude intersects the base P. The Law of Sines tells us that the length from P to any corner is $4/\sqrt{3}$. To see, why consider that the base of the tetrahedron looks like so:



By the Law of Sines,

$$\frac{\sin 120^\circ}{4} = \frac{\sin 30^\circ}{a} \quad\Longrightarrow\quad \frac{4}{\sin 120^\circ} = \frac{a}{\sin 30^\circ} \quad\Longrightarrow\quad \frac{4}{\sqrt{3}/2} = \frac{a}{1/2} \quad\Longrightarrow\quad \frac{4}{\sqrt{3}} = a.$$

Now look at the triangle formed by a side of the tetrahedron of length 4, the altitude of length h, and any of those lines of length $a = 4/\sqrt{3}$. The Pythagorean Theorem tells us that $a^2 + h^2 = 4^2$, so $h = \sqrt{32/3} = 4\sqrt{2}/\sqrt{3}$.

To express s as a function of x, observe that when $x = h = 4\sqrt{2}/\sqrt{3}$, the side length s is 4. This gives us a proportion that we can solve for s:

$$\frac{x}{4\sqrt{2}/\sqrt{3}} = \frac{s}{4} \quad\Longrightarrow\quad s = \frac{x\sqrt{3}}{\sqrt{2}}.$$

We can now substitute for s and h in our formula for volume:

$$\begin{aligned}
V &= \int_0^h \frac{s^2\sqrt{3}}{4}\,dx \\
&= \int_0^h \left(\frac{\sqrt{3}}{\sqrt{2}} \cdot x\right)^2 \cdot \frac{\sqrt{3}}{4}\,dx \\
&= \frac{3\sqrt{3}}{8} \int_0^h x^2\,dx \\
&= \frac{3\sqrt{3}}{8} \cdot \frac{x^3}{3}\bigg|_0^h \\
&= \frac{\cancel{3}\sqrt{3}}{8} \cdot \left(\frac{\left(\frac{4\sqrt{2}}{\sqrt{3}}\right)^3}{\cancel{3}} - 0\right) \\
&= \frac{\cancel{\sqrt{3}}}{\cancel{8}} \cdot \frac{\cancelto{8}{64} \cdot 2\sqrt{2}}{3\cancel{\sqrt{3}}} \\
&= \frac{16\sqrt{2}}{3}.
\end{aligned}$$