HOMEWORK QUIZ 5 SOLUTIONS

p. 408

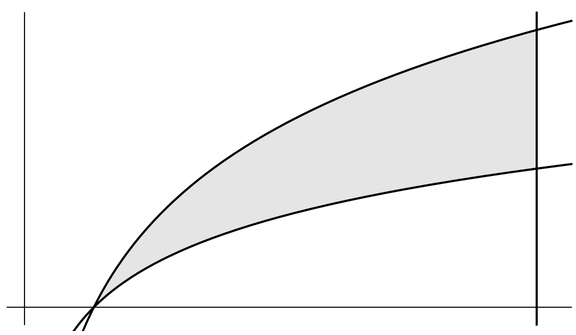
13. First we need to find the points of intersection:

$$\begin{aligned} 1 &= \frac{2}{1+x^2} \\ 1+x^2 &= 2 \\ x^2-1 &= 0 \end{aligned}$$

so $x = \pm 1$. We can verify that $2/1+x^2$ is on top (since $2/1+0^2 = 2 > 1$) so the area is

$$\begin{aligned} \int_{-1}^{1} \frac{2}{1+x^2} - 1\,dx &= 2\arctan x - x\Big|_{-1}^{1} \\ &= (2\arctan 1 - 1) - (2\arctan(-1) - (-1)) \\ &= 2 \times \frac{\pi}{4} - 1 - 2 \times \left(-\frac{\pi}{4}\right) - 1 \\ &= \pi - 2. \end{aligned}$$

26. The graph looks like this:



The hint directed you to integrate with respect to y. First, find the points of intersection *as y-values.* We can find the intersections of $x = e^2$ with $y = \ln x$ and $y = \ln x^2$ by substitution:

$$\ln e^2 = 2 \qquad \ln e^4 = 4$$

but the intersection of $y = \ln x$ and $y = \ln x^2$ requires a little more work:

$$\begin{aligned} \ln x^2 &= \ln x \\ x^2 &= x \\ x^2 - x &= 0 \\ x(x-1) &= 0. \end{aligned}$$

Notice that we can't actually have $x = 0$, since $\ln x$ and $\ln x^2$ are undefined there. So the third point of intersection is $x = 1$. We need the y-value, which is easy enough: $\ln 1 = \ln 1^2 = 0$.

Again, we have to solve for the functions for x. The function $x = e^2$ is already there; the other two functions require us to solve in terms of y:

$$\begin{aligned} y &= \ln x & y &= \ln x^2 \\ e^y &= x & e^y &= x^2 \\ & & \sqrt{e^y} &= x \\ & & e^{y/2} &= x. \end{aligned}$$

Hence the area is

$$\begin{aligned} \int_0^2 e^y - e^{y/2}\,dy + \int_2^4 e^2 - e^{y/2} &= \left(e^y - \frac{e^{y/2}}{1/2}\right)\Bigg|_0^2 + \left(e^2 y - \frac{e^{y/2}}{1/2}\right)\Bigg|_2^4 \\ &= \left[\left(e^2 \cancel{-2e^1}\right) - \left(e^0 - 2e^0\right)\right] + \left[\left(4e^2 - 2e^2\right) - \left(2e^2 \cancel{-2e^1}\right)\right] \\ &= e^2 - (-1) \\ &= e^2 + 1. \end{aligned}$$

51. Before doing either (a), (b), or (c), it would save time to compute a formula for the areas of the regions once, then apply it to each case. The points of intersection for R_1 are

$$x = x^2 \quad\Longrightarrow\quad x^2 - x = 0 \quad\Longrightarrow\quad x(x-1) = 0 \quad\Longrightarrow x = 0, 1.$$

The points of intersection for R_2 are the same, and are found similarly.

To compute the area, notice that $x^p < x < x^{1/q}$ (you can verify this by substituting $x = 1/2$ into each function). Hence the area of R_1 is

$$A_1 = \int_0^1 x - x^p\,dx = \left(\frac{x^2}{2} - \frac{x^p}{p+1}\right)\Bigg|_0^1 = \frac{1}{2} - \frac{1}{p+1} = \frac{(p+1)-2}{2(p+1)} = \frac{p-1}{2(p+1)}$$

and the area of R_2 is

$$A_2 = \int_0^1 x^{1/q}\,dx = \left(\frac{x^{1/q}}{1/q + 1} - \frac{x^2}{2}\right)\Bigg|_0^1 = \left(\frac{qx^{1/q}}{1+q} - \frac{x^2}{2}\right)\Bigg|_0^1 = \frac{q}{q+1} - \frac{1}{2} = \frac{q-1}{2(q+1)}.$$

(a) When $p = q$,

$$A_1 = \frac{p-1}{2(p+1)} = A_2.$$

(b) When $p > q$, say $p = q + k$ for some positive integer k, then

$$A_1 = \frac{p-1}{2(p+1)} = \frac{(q+k)-1}{2((q+k)+1)}.$$

while

$$A_2 = \frac{q-1}{2(q+1)}.$$

To compare them, find a common denominator

$$A_1 = \frac{[(q+k)-1][q+1]}{2(q+1)[(q+k)+1]} \qquad \text{and} \qquad A_2 = \frac{(q-1)[(q+k)+1]}{2(q+1)[(q+k)+1]}$$

then compare the numerators

$$\text{num}\,(A_1) = q^2 \cancel{+q} + qk + k \cancel{-q} - 1 \qquad \text{and} \qquad \text{num}\,(A_2) = q^2 + qk \cancel{+q} \cancel{-q} - k - 1.$$

Notice that the first has $+k$, while the second has $-k$; everything else is the same. Since k is positive, $+k > 0 > -k$, so $A_1 > A_2$.

(c) The analysis is the same as in (b), except that k is a negative integer. Hence $+k < 0 < -k$, so $A_1 < A_2$.

p. 420

15. We can view a tetrahedron as an accumulation of squares:

$$\int_0^h x^2\,dx = \left.\frac{x^3}{3}\right|_0^h = \frac{h^3}{3}.$$

But what is h? It is the length of an altitude that we drop from the tip of the tetrahedron to its base. Call the point where this altitude intersects the base P. The Pythagorean Theorem tells us that the length from P to any corner is $2\sqrt{2}$. This line segment forms, with the altitude and a side of the tetrahedron from said corner to tip, another right triangle; the Pythagorean Theorem tells us that the length of the altitude is also $2\sqrt{2}$. Hence $h = 2\sqrt{2}$ and the volume of the tetrahedron is

$$\frac{\left(2\sqrt{2}\right)^3}{3} = \frac{16\sqrt{2}}{3}.$$

29. Applying the washer method ($\pi \int \text{top}^2\,dx - \pi \int \text{bottom}^2\,dx$), we have

$$\begin{aligned} V &= \pi \int_{\ln 2}^{\ln 3} e^x\,dx - \pi \int_{\ln 2}^{\ln 3} e^{-x}\,dx \\ &= \pi \left[e^x\big|_{\ln 2}^{\ln 3} + e^{-x}\big|_{\ln 2}^{\ln 3} \right] \\ &= \pi \left[\left(e^{\ln 3} - e^{\ln 2}\right) + \left(e^{-\ln 3} - e^{-\ln 2}\right) \right] \\ &= \pi \left[(3-2) + \left(e^{\ln \frac{1}{3}} - e^{\ln \frac{1}{2}}\right) \right] \\ &= \pi \left[1 + \left(\frac{1}{3} - \frac{1}{2}\right) \right] \\ &= \pi \left[1 + \left(-\frac{1}{6}\right) \right] \\ &= \frac{5\pi}{6}. \end{aligned}$$

35. Applying the washer method using Δy as the width of a rectangle, we have

$$
\begin{aligned}
V &= \pi \int_0^6 y^2\,dy - \pi \int_0^6 \left(\frac{y}{2}\right)^2\,dy \\
&= \pi \left(\frac{y^3}{3}\bigg|_0^6 - \frac{y^3}{12}\bigg|_0^6 \right) \\
&= \pi \left[\left(\frac{6^3}{3} - 0 \right) - \left(\frac{6^3}{12} - 0 \right) \right] \\
&= \pi\,(72 - 18) \\
&= 54\pi.
\end{aligned}
$$