HOMEWORK QUIZ 3 SOLUTIONS

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23. $\int_0^1 x^2 - 2x + 3\,dx = {}^{x^3}\!/_3 - x^2 + 3x\Big|_0^1 = {}^1\!/_3 - 1 + 3 = {}^7\!/_3$. This seems consistent, because the area should be more than the rectangle, which has area 2, but only by a little, since the triangle enclosing the area above the rectangle would have area ${}^1\!/_2 \cdot 1 \cdot 1 = {}^1\!/_2$.

39. $\int_0^{\pi/4} \sec^2\theta\,d\theta = \tan\theta\Big|_0^{\pi/4} = 1 - 0 = 1.$

45. $\int_1^2 {}^3\!/_t\,dt = 3\ln t\Big|_1^2 = 3\ln 2 - 3\ln 0 = 3\ln 2.$

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15. Let $u = \sin x$. Since $u' = \cos x$, we have $\int \sin^3 x \cos x\,dx = \int u^3 u'\,dx = \int u^3\,du = {}^{u^4}\!/_4 + C = {}^{\sin^4 x}\!/_4 + C$.

19. Let $u = 1 - 4x^3$. Since $u' = -12x^2$, we have

$$\begin{aligned}
\int \frac{2}{\sqrt{1-4x^3}}\,dx &= 2 \cdot \underbrace{\left(-\frac{1}{12}\right)}_{\text{to balance } u'} \int \frac{1}{\sqrt{1-4x^3}} \cdot \underbrace{(-12x^2)}_{u'}\,dx \\
&= -\frac{1}{6}\int \frac{1}{\sqrt{u}} u'\,dx \\
&= -\frac{1}{6}\int \frac{1}{\sqrt{u}}\,du \\
&= -\frac{1}{6}\int u^{-1/2}\,du \\
&= -\frac{1}{6} \cdot \left(\frac{u^{1/2}}{1/2} + C\right) \\
&= -\frac{1}{3}\sqrt{1-4x^3} + C.
\end{aligned}$$

29. Let $u = 2x$. Since $u' = 2$, we have

$$\int \frac{dx}{1+4x^2} = \underbrace{\frac{1}{2}}_{\text{to balance } u'} \int \frac{1}{1+u^2}\underbrace{2}_{u'}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1} 2x + C.$$

35. Let $u = x + 4$. Since $x = u - 4$ and $u' = 1$, we have

$$\begin{aligned}\int \frac{x}{\sqrt[3]{x+4}}\,dx &= \int \frac{u-4}{\sqrt[3]{u}}u'\,dx \\ &= \int \frac{u}{\sqrt[3]{u}} - \frac{4}{\sqrt[3]{u}}\,du \\ &= \int u^{2/3} - 4u^{-1/3}\,du \\ &= \frac{u^{5/3}}{5/3} - 4\cdot\frac{u^{2/3}}{2/3} \\ &= \frac{3\sqrt[3]{(x+4)^5}}{5} - 6\sqrt[3]{(x+4)^2}.\end{aligned}$$

This can be written in a form similar to the book's, but I'm having none of that here. See me if you want to see how to do that.

51. Let $u = 3x$. Since $u' = 3$, we have

$$\begin{aligned}\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2+1}\,dx &= 4\cdot\underbrace{\frac{1}{3}}_{\text{to balance } u'}\int_{x=1/3}^{1/\sqrt{3}} \frac{1}{u^2+1}\cdot\underbrace{3}_{u'}\,dx \\ &= \frac{4}{3}\int_{x=1/3}^{1/\sqrt{3}} \frac{1}{u^2+1}\,du \\ &= \frac{4}{3}\tan^{-1} u\Big|_{x=1/3}^{1/\sqrt{3}} \\ &= \frac{4}{3}\tan^{-1} 3x\Big|_{x=1/3}^{1/\sqrt{3}} \\ &= \frac{4}{3}\left(\tan^{-1}\frac{3}{\sqrt{3}} - \tan^{-1} 1\right) \\ &= \frac{4}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{12}.\end{aligned}$$

53. We have to use the half-angle formula here.

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2 x\,dx &= \int_{-\pi}^{\pi} \frac{1+\cos 2x}{2}\,dx\\ &= \frac{1}{2}\int_{-\pi}^{\pi} 1+\cos 2x\,dx\\ &= \frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)\Big|_{-\pi}^{\pi}\\ &= \frac{1}{2}\left[(\pi+0)-(-\pi+0)\right]\\ &= \frac{2\pi}{2}=\pi.\end{aligned}$$

77. The denominator factors as $(3x+1)^2$. Let $u=3x+1$. Since $u'=3$, we have

$$\begin{aligned}\int_1^2 \frac{4}{9x^2+6x+1}\,dx &= \int_1^2 \frac{4}{(3x+1)^2}\,dx\\ &= 4\cdot \underbrace{\frac{1}{3}}_{\text{to balance } u'} \int_{x=1}^2 \frac{1}{u^2}\cdot \underbrace{3}_{u'}\,dx\\ &= \frac{4}{3}\int_{x=1}^2 u^{-2}\,du\\ &= \frac{4}{3}\cdot -u^{-1}\Big|_{x=1}^2\\ &= -\frac{4}{3}\cdot\frac{1}{3x+1}\Big|_1^2\\ &= -\frac{4}{3}\left(\frac{1}{7}-\frac{1}{4}\right)\\ &= \frac{1}{7}.\end{aligned}$$

93. The area of the first diagram is

$$A_1=\int_4^9 \frac{(\sqrt{x}-1)^2}{2\sqrt{x}}\,dx.$$

To evaluate it, we could let $u=\sqrt{x}-1$. Since $u'={}^1\!/_{2\sqrt{x}}$, we have

$$A_1=\int_{x=4}^9 u^2u'\,dx=\int_{x=4}^9 u^2\,du.$$

We could also replace the limits of integration by their corresponding values of u. Since $u=\sqrt{x}-1$, we have $u(4)=1$ and $u(9)=2$. That gives us

$$A_1=\int_1^2 u^2\,du.$$

This is almost identical to the area of the second diagram,

$$A_2 = \int_1^2 x^2\,dx.$$

The only difference is the variable name, x or u. But, as Shakespeare put it,

> What's in a name? that which we call a rose
> By any other name would smell as sweet;
> So Romeo would, were he not Romeo call'd,
> Retain that dear perfection which he owes
> Without that title. Romeo, doff thy name;
> And for that name, which is no part of thee,
> Take all myself.

So we can doff the u, put on (a different) x, and take Juli... er, the area itself.

102. Let $u = ax$. Since $u' = a$, we have

$$\begin{aligned}
\int \sin^2 ax\,dx &= \underbrace{\frac{1}{a}}_{\text{to balance } u'} \int \sin^2 u \cdot \underbrace{a}_{u'}\,dx \\
&= \frac{1}{a}\int \sin^2 u\,u'\,dx \\
&= \frac{1}{a}\int \sin^2 u\,du \\
&= \frac{1}{a}\int \frac{1-\cos 2u}{2}\,du \\
&= \frac{1}{2a}\left(u - \frac{\sin 2u}{2} + C\right) \\
&= \frac{u}{2a} - \frac{\sin 2u}{4a} + C \\
&= \frac{x}{2} - \frac{\sin(2ax)}{4a} + C,
\end{aligned}$$

and

$$\begin{aligned}
\int \cos^2 ax\,dx &= \underbrace{\frac{1}{a}}_{\text{to balance } u'} \int \cos^2 u \cdot \underbrace{a}_{u'}\,dx \\
&= \frac{1}{a}\int \cos^2 u\, u'\,dx \\
&= \frac{1}{a}\int \cos^2 u\,du \\
&= \frac{1}{a}\int \frac{1+\cos 2u}{2}\,du \\
&= \frac{1}{2a}\left(u + \frac{\sin 2u}{2} + C\right) \\
&= \frac{u}{2a} + \frac{\sin 2u}{4a} + C \\
&= \frac{x}{2} + \frac{\sin(2ax)}{4a} + C.
\end{aligned}$$