HOMEWORK QUIZ 1 SOLUTIONS

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6. An example of a limit whose form as $x \to 0$ is ∞/∞ is $\lim_{x\to 0} \ln x/(1/x)$.

17. Don't forget that ln only applies to one symbol on its right, unless parentheses are involved.

$$\lim_{x\to e} \frac{\ln x - 1}{x - e} \begin{matrix}\nearrow \ln e - 1 = 0 \\ \searrow e - e = 0\end{matrix} \overset{\text{L'H}}{=} \lim_{x\to e} \frac{\frac{1}{x} - 0}{1 - 0} = \frac{1/e}{1} = \frac{1}{e}.$$

25. Don't forget the Chain Rule when differentiating the denominator.

$$\lim_{x\to\pi} \frac{\cos x + 1}{(x-\pi)^2} \begin{matrix}\nearrow \cos\pi + 1 = 0 \\ \searrow (\pi-\pi)^2 = 0\end{matrix} \overset{\text{L'H}}{=} \lim_{x\to\pi} \frac{-\sin x + 0}{2(x-\pi)\cdot(1-0)} \begin{matrix}\nearrow -\sin\pi = 0 \\ \searrow 2(\pi-\pi) = 0\end{matrix} \overset{\text{L'H}}{=} \lim_{x\to\pi} \frac{-\cos x}{2\cdot 1} = \frac{-\cos\pi}{2} = \frac{1}{2}.$$

53.

$$\lim_{\theta\to\frac{\pi}{2}^-} (\tan\theta - \sec\theta) \to \infty - \infty$$

$$\lim_{\theta\to\frac{\pi}{2}^-} \left(\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}\right)$$

$$\lim_{\theta\to\frac{\pi}{2}^-} \left(\frac{\sin\theta - 1}{\cos\theta}\right) \begin{matrix}\nearrow \sin\frac{\pi}{2} - 1 = 0 \\ \searrow \cos\frac{\pi}{2} = 0\end{matrix}$$

$$\lim_{\theta\to\frac{\pi}{2}^-} \frac{\cos\theta}{-\sin\theta} \begin{matrix}\nearrow \cos\frac{\pi}{2} = 0 \\ \searrow -\sin\frac{\pi}{2} = -1\end{matrix}$$

$$\frac{0}{-1}$$

$$0$$

69. This is covered in the textbook on pp. 297–299. We compute

$$\lim_{x\to\infty} \frac{x^{10}}{e^{.01x}} \begin{matrix}\nearrow \infty \\ \searrow \infty\end{matrix} = \lim_{x\to\infty} \frac{10x^9}{.01e^{.01x}} \begin{matrix}\nearrow \infty \\ \searrow \infty\end{matrix} = \lim_{x\to\infty} \frac{10\cdot 9x^8}{(.01)^2 e^{.01x}} \begin{matrix}\nearrow \infty \\ \searrow \infty\end{matrix} = \cdots = \lim_{x\to\infty} \frac{10!}{(.01)^{10} e^{.01x}} \begin{matrix}\nearrow 10! \\ \searrow \infty\end{matrix} \to 0.$$

When the limit of the ratios of the functions is 0, the denominator grows much faster than the numerator. Hence $e^{.01x}$ grows much faters than x^{10}.

Remark: When we write an excalamation point to the right of a number, it means to multiply that number to all the positive integers below it. In this case, $10! = 10 \times 9 \times 8 \times 7 \times \cdots \times 2 \times 1$.

103. (a) Compound interest pays (or charges) interest on the *balance in the account* at the end of each compounding period. Here,

$$\begin{aligned}
B(0) &= \underbrace{P}_{\text{initial deposit}} . \\
B(1) &= \underbrace{P}_{\text{initial deposit}} + \underbrace{Pr}_{\text{interest on balance}} \\
&= P(1+r). \\
B(2) &= \underbrace{P(1+r)}_{\text{amount in account after 1 yr}} + \underbrace{[P(1+r)]\,r}_{\text{interest on balance}} \\
&= \underbrace{P(1+r)}_{\text{common factor}} \times \left(\underbrace{1}_{\frac{P(1+r)}{P(1+r)}} + \underbrace{r}_{\frac{[P(1+r)]r}{P(1+r)}} \right) \\
&= P(1+r)^2 \\
&\vdots \\
B(t) &= \underbrace{P(1+r)^{t-1}}_{\text{amount in account after } t-1 \text{ years}} + \underbrace{\left[P(1+r)^{t-1}\right]}_{\text{interest on balance}} r \\
&= \underbrace{P(1+r)^{t-1}}_{\text{common factor}} \times \left(\underbrace{1}_{\frac{P(1+r)^{t-1}}{P(1+r)^{t-1}}} + \underbrace{r}_{\frac{[P(1+r)^{t-1}]r}{P(1+r)^{t-1}}} \right) \\
&= P(1+r)^{t-1}.
\end{aligned}$$

(b) Since

$$\lim_{m\to\infty} P\left(1+\frac{r}{m}\right)^{mt} = P \cdot \lim_{m\to\infty} \left(1+\frac{r}{m}\right)^{mt} \to P\cdot(1+0)^{\infty},$$

we see the indeterminate form 1^∞ and realize we need to rewrite the experssion using a logarithm. Let $y = (1+{}^r\!/\!_m)^{mt}$ (the problematic part of the limit); by properties of logarithms,

$$\ln y = \ln(1+{}^r\!/\!_m)^{mt} = (mt)\ln(1+{}^r\!/\!_m),$$

so instead of $\ln y$ we consider

$$\lim_{m\to\infty} \ln y = \lim_{m\to\infty} (mt)\ln(1+{}^r\!/\!_m) \to m\cdot\infty\cdot\ln(1+0) \to m\cdot\infty\cdot\ln 1 \to m\cdot\infty\cdot 0.$$

We cannot apply L'Hôpital's Rule directly to this expression; we need to rewrite it so that the limit approaches ${}^0\!/\!_0$ or ${}^{\pm\infty}\!/\!_{\pm\infty}$. In this case, the easier way to accmplish this is to write

$$\lim_{m\to\infty} \frac{\ln(1+{}^r\!/\!_m)}{{}^1\!/\!_{mt}} \begin{matrix}\nearrow \ln(1+0)\to 0 \\ \searrow {}^1\!/\!_\infty \to 0\end{matrix} \overset{\text{L'H}}{=\!=} \lim_{m\to\infty} \frac{\frac{1}{1+{}^r\!/\!_m}\cdot\left(-{}^r\!/\!_{m^2}\right)}{-{}^1\!/\!_{m^2}\cdot{}^1\!/\!_t} = \lim_{m\to\infty}\left(rt\times\frac{1}{1+{}^r\!/\!_m}\right) \to rt\times\frac{1}{1+0} = rt.$$

This is the limit of $\ln y$. Substituting into its definition, we have

$$\lim_{m\to\infty}\ln y = rt \quad\Rightarrow\quad \ln\left(\lim_{m\to\infty} y\right) = rt \quad\Rightarrow\quad \lim_{m\to\infty} y = e^{rt}.$$

We wanted $\lim_{m\to\infty} y$ to start with; combining with the P that we omitted while we worked on the limit, we have

$$\lim_{m\to\infty} P\left(1+\frac{r}{m}\right)^{mt} = Pe^{rt},$$

as claimed.

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1. Newton's method approximates roots of a function using roots of tangents lines. Near the point of intersection, a tangent line travels in roughly the same direction as the curve. So, if we compute a tangent line near the actual root, the tangent line's root should approximate the actual root.

5. Use

$$x_{i+1} = -\frac{f(x_i)}{f'(x_i)} + x_i = -\frac{x_i^2-6}{2x_i} + x_i.$$

If we start at $x_0 = 3$, this gives us the successive approximations $x_1 = 2.5$ and $x_2 = 2.45$, which rounds to 2.50. We can therefore stop the algorithm.

15. For the root of $f(x) = \sin x - {}^{x}/_{2}$, use

$$x_{i+1} = -\frac{f(x_i)}{f'(x_i)} + x_i = -\frac{\sin x_i - {}^{x}/_{2}}{\cos x_i - {}^{1}/_{2}}.$$

A glance at a graph suggests that $x = 2$ is close to the root. Starting from there, we obtain the approximations

$$x_0 = 2, \quad x_1 = 1.90, \quad x_2 = 1.90,$$

so we take $x \approx 1.90$ as one root. A second glance at the graph suggests $x = 0$ is close to the root. In fact, $x = 0$ *is* a root, as we discover if we try to use Newton's method. A third glance shows that $x = -2$ is close to the root. While skipping the details, we point out that this gives

$$x_0 = -2, \quad x_1 = -1.90, \quad x_2 = -1.90.$$

So the there roots are approximately $x = 0$, $x = 1.90$, and $x = -1.90$.

27. (a) *True:* We approximate using roots of tangent lines. The tangent lines are themselves approximations to the curve at the points of intersection.
(b) *False:* The quadratic formula gives *exact* answers, but Newton's Method provides only *approximate* answers.
(c) *False:* Newton's method can find the exact root in certain circumstance; for example, when $f(x) = mx + b$ for non-zero integers m and b.